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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 64 NUMBER 4

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The AMERICAN MATHEMATICAL MONTHLY

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $6.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y., during the months of January, February, March, April, May, June-July, August-September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

COMMUTATIVITY IN FINITE MATRICES*

OLGA TAUSSKY, National Bureau of Standards

1. Introduction. The real and complex numbers have the properties

(1) $ab = ba$ for all a and b

and

(2) for every $a \neq 0$ there is an inverse a^{-1},

which implies

(2′) $ab = 0$ only if $a = 0$ or $b = 0$.

A classical theorem of Frobenius† states that there are no other hypercomplex systems which have these properties. If we want to consider more general systems, we must give up at least one of these properties. For example, if we do not insist on the commutative law (1), the quaternions of Hamilton become acceptable, but no others. The set of $n \times n$ matrices $A, B, \cdots$, with complex elements can be regarded as a hypercomplex system with n^2 base elements and in this system, in general, $AB \neq BA$. In general, also, there is no inverse A^{-1} even if $A \neq 0$ and, furthermore, AB can be zero without $A = 0$ or $B = 0$. The two axioms may even be violated simultaneously, so that when $AB = 0$ we may still have $BA \neq 0$. A simple example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As soon as $AB \neq BA$ for a pair A, B, various other things go wrong too. Of these, we mention three:

(a) If $A_n = (a_{ik}^{(n)})$ and we define $\lim_{n\to\infty} A_n$ to be $(\lim_{n\to\infty} a_{ik}^{(n)})$, we may define the exponential of a matrix X by the formal power series $e^X = \sum_{i=0}^{\infty} X^i/i!$. Then, in general, $e^A e^B \neq e^{A+B}$, but when $AB = BA$, we always have $e^A e^B = e^{A+B}$.

Example: If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^2 = 0, \quad B^2 = 0, \quad (A + B)^2 = I,$$

---

* Invited address at the meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 5, 1956.

The preparation of this paper was supported (in part) by the Office of Naval Research.

† References listed by sections will be found at the end of the paper.

$$e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad e^B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

but

$$e^{A+B} = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}.$$

(b) Let $\lim_{n\to\infty} A^n = X$, and $\lim_{n\to\infty} B^n = Y$. Then $\lim_{n\to\infty} (AB)^n$ does not exist, in general, when $AB \neq BA$, while for $AB = BA$ we have $(AB)^n = A^n B^n$ for all $n \geq 1$, so that $\lim_{n\to\infty} (AB)^n = \lim_{n\to\infty} A^n \lim_{n\to\infty} B^n = XY$.

Examples: If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so that $X = 0$, $Y = 0$, but

$$\lim_{n\to\infty} (AB)^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq XY.$$

If

$$A = \begin{pmatrix} 1/2 & 2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 & 0 \\ 2 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 4 + 1/4 & 0 \\ 0 & 0 \end{pmatrix},$$

and $X = 0$, $Y = 0$, but $\lim_{n\to\infty} (AB)^n$ does not exist.

(c) Let the eigenvalues of A be $\alpha_1, \cdots, \alpha_n$ and those of B, $\beta_1, \cdots, \beta_n$. When $AB \neq BA$, the eigenvalues $\gamma_1, \cdots, \gamma_n$ of AB are, in general, not $\alpha_1\beta_1, \cdots, \alpha_n\beta_n$ for any pairing. When $AB = BA$, however, the eigenvalues of any polynomial $p(A, B)$ are $p(\alpha_i, \beta_i)$, by another classical theorem of Frobenius.

Example: If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and $\alpha_1=\alpha_2=0$, $\beta_1=\beta_2=0$, while $\gamma_1=1$, $\gamma_2=0$.

The main object of this paper is a study of the replacement of $AB=BA$ in (c) by a weaker rule.

2. Commutators. Two theorems of McCoy and Drazin that are stated in Section 3 involve the notion of the commutator of two matrices A and B. We call $AB-BA$ the commutator of A and B and denote it by (A, B). Since $(A, B)=0$ when $AB=BA$, the commutator may be said to "measure" how much AB differs from BA. In the theory of groups, if $AB\neq BA$, then $A^{-1}B^{-1}AB$ is called the commutator of A and B. Since, for matrices, A^{-1} and B^{-1} do not always exist, but addition and multiplication are defined, we take instead, $AB-BA$. We can also study commutators of higher order, for example, $(A, (A, B))=A(AB-BA)-(AB-BA)A$ or $(B, (A, B))$.

The following remarks, about a special case of commutators, while not immediately relevant to our topic, indicate an interesting area of study.

Consider the commutators in which the last factor is A^*, the others all A, where A^* is the transposed and conjugate complex matrix of A. We write $(A, A^*)=AA^*-A^*A=C_2$, $A(A, A^*)-(A, A^*)A=C_3$, *etc.* Matrices for which $C_2=(A, A^*)=0$, the so-called normal matrices, play an important role as a natural generalization of hermitian matrices.

Suppose we have a matrix for which $C_2\neq 0$, we might ask whether or not $C_3=0$. It turns out that if $C_2\neq 0$ then $C_3\neq 0$. Further if, $C_2\neq 0$ then $(C_2, C_3)\neq 0$. Let us next consider C_4. We find that $C_4=0$ is possible even if $C_2\neq 0$, $C_3\neq 0$.

Example: If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then

$$A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and we have

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad C_4 = 0, \quad (C_2, C_3) = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}.$$

However, if in the case $n=2$, we have $C_2\neq 0$, $C_3\neq 0$, $C_4\neq 0$, then $C_n\neq 0$ for any n. For $n=3$ the behavior is different as is shown by the following examples:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_4 \neq 0, \quad C_5 = 0; \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_5 \neq 0, \quad C_6 = 0.$$

3. Two theorems of McCoy and Drazin. McCoy proved that if $(A, B) \neq 0$, but $(A, (A, B)) = (B, (A, B)) = 0$, then, although $AB \neq BA$, all eigenvalues of the product AB are of the form $\alpha_i\beta_i$ for a certain pairing and, more generally, all polynomials $p(A, B)$ have eigenvalues $p(\alpha_i, \beta_i)$ for the same pairing.

Later, M. P. Drazin generalized McCoy's theorem by showing that, if all commutators of A and B of a fixed order k (*i.e.*, involving k brackets) vanish, then all polynomials $p(A, B)$ have eigenvalues $p(\alpha_i, \beta_i)$.

Apparently there are none or at most very few theorems known which say: "such and such a statement is true if and only if $AB = BA$"; it is always "if $AB = BA$ then . . . ". Hence we are tempted to replace $AB = BA$ in various ways by weaker hypotheses. One way is to assume that instead of $AB - BA = 0$ some of the higher commutators vanish. Another one is suggested by McCoy and Drazin's theorems: It is known that $AB = BA$ implies that all $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$, but the converse is not true. Consider, for example,

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Here $AB \neq BA$, but it can be shown (*cf.* Condition 4.1 below) that all $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$. Further, no commutator of the form $(A, (A, (A, \cdots (A, (A, B)) \cdots)))$, however long, can vanish.

4. Property *P*. McCoy asked and answered the question: which are the pairs of $n \times n$ matrices A, B, (with complex elements) for which all polynomials $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$? Each of the following two conditions is necessary and sufficient.

Condition 4.1. *There exists a matrix S, such that $S^{-1}AS$, $S^{-1}BS$ are both left triangular, or both right triangular.*

Condition 4.2. *All matrices $f(A, B)(AB - BA)$ are nilpotent, where f is an arbitrary polynomial in A and B, and nilpotency of a matrix X means that $X^r = 0$ for a certain integer r.*

A special case of McCoy's theorem was pointed out recently by H. Schneider, namely, if $AB = 0$ and $BA \neq 0$. Then $AB - BA = -BA$ and $(-BA)^2 = 0$, further $A(BA) = 0$, $(B(BA))^2 = 0$. Hence, by Condition 4.2, such a pair qualifies, and from Condition 4.1 it follows that a matrix S exists such that $S^{-1}AS$, $S^{-1}BS$ are both triangular.

Another example of pairs of matrices with property P was used by A. Brauer. Let A be any $n \times n$ matrix and v one of its eigenvectors. Let B be an $n \times n$ matrix of rank 1 whose columns are all multiples of v. Then $n - 1$ of the eigenvalues of $A + B$ coincide with $n - 1$ eigenvalues of A. One reason for this is that B has $n - 1$ eigenvalues zero. Further it can be shown that A and B have property P. For, let SAS^{-1} be triangular with the eigenvalue corresponding to v in the upper left corner. Then v goes over into Sv which is the vector $(1, 0, \cdots, 0)$. Hence SBS^{-1} is triangular too, with $n - 1$ zeros in the main diagonal.

5. Property *L*. We obtain a larger class of matrices if we do not assume that all polynomials $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$, but only that a subset has this property. We may assume, for example, that all $\lambda A + \mu B$ have as eigenvalues $\lambda\alpha_i + \mu\beta_i$, where λ, μ are arbitrary, complex numbers. Such a pair A, B is said to have property L. We now obtain a larger class of matrices if $n \geq 3$ (for $n = 2$ the classes coincide). For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 1 & 2 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

have eigenvalues, 1, 2, 3, and -2, -2, 1. However, AB does not have the eigenvalues 1 or -2. On the other hand, it can be shown that the pair A, B has property L. Another example is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In this case all matrices $\lambda A + \mu B$ have all eigenvalues 0, but

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has some eigenvalues $\neq 0$.

If, however, A and B are both normal, then it is true that $AB = BA$ if we assume only that $\lambda A + \mu B$ have all eigenvalues $\lambda\alpha_i + \mu\beta_i$. This was proved by Wiegmann and Wielandt. Wielandt proved it from the following more general theorem:

Let A, B be normal and let $\gamma_i(z)$ be the eigenvalues of $A + zB$ and assume that $\sum_{i=1}^{n} |\gamma_i(z)|^2 \geq \sum_{i=1}^{n} |\alpha_i + z\beta_i|^2$ for at least three values of z which are the vertices of a triangle in the z-plane which contains O in the interior. It follows that $AB = BA$.

We, of course, assume much more, namely, that $\gamma_i(z) = \alpha_i + z\beta_i$ for all values of z.

6. Matrices in $A + zB$ with multiple eigenvalues. In the previous section we treated the matrices for which all $\lambda A + \mu B$ have eigenvalues $\lambda\alpha_i + \mu\beta_i$, and again, we can point out a larger class.

This can be done in a way that is easily described for $n = 2$. Here a pencil of matrices $A + zB$ has either all matrices with a double eigenvalue, or exactly two, or exactly one. There is no other possibility. If the pair A, B has property L, then it is very easy to see that either all matrices have a double eigenvalue, or exactly one does. However, the converse of this is true too. For $n > 2$ either all matrices in the pencil have a multiple eigenvalue or at most $n(n-1)$ do. If, however, the pair A, B has property L, then either all matrices in the pencil

$A+zB$ have a multiple eigenvalue or at most $n(n-1)/2$ do. The converse of this is not true, as is seen by the pair

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix}.$$

Here both A and B have a triple eigenvalue 0, but $A+B$ is nonsingular, so that property L does not hold. On the other hand, the matrices A and B are the only ones in the whole pencil $\lambda A+\mu B$ which have a multiple eigenvalue. However, a partial converse is true even for $n>2$.

7. Diagonable pencils. In a discussion of multiple eigenvalues we must consider eigenvectors too. We know, for example, that if all matrices $A+zB$ have the full number of eigenvectors (*i.e.*, if for all finite values of z and $z=\infty$, the matrix $A+zB$ is similar to a diagonal matrix), then $AB=BA$. If, in the whole pencil, even a single matrix is not similar to a diagonal matrix then AB need not be equal to BA. This is shown by the example,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, if $AB=BA$, then the pencil $A+zB$ contains either only diagonable matrices, or none, or exactly one. For $n=2$ the case of "none" never happens. For $n=3$ an example of a pencil with no diagonable matrices apart from 0 is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A pencil with only one diagonable matrix is given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \beta & \beta_1 \\ 0 & \beta \end{pmatrix}.$$

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POWER SERIES EXPANSIONS FOR INVERSE FUNCTIONS

J. S. FRAME, Michigan State University

1. Introduction. Numerical values for all the roots of a trinomial equation such as

$$z^3 - 3z + 1 = 0, \tag{1.1}$$

may be obtained from convergent "inverted" power series. To solve (1.1) we may rewrite it in the following forms,

$$w^2 = z^2(1 + z^{-3}), \qquad \text{where } w^2 = 3, \tag{1.2a}$$

$$w = z(1 - z^2/3), \qquad \text{where } w = 3^{-1}, \tag{1.2b}$$

and then invert each of these by a method to be explained below which yields the "inverted" power series

$$\begin{aligned} z &= w \sum_{k=0}^{\infty} \frac{1}{1-3k} \binom{(3k-1)/2}{k} w^{-3k} \\ &= w - 2^{-1}w^{-2} - 3\cdot 2^{-3}w^{-5} - 2^{-1}w^{-8} - 105\cdot 2^{-7}w^{-11} - 3\cdot 2^{-1}w^{-14} \\ &\quad - 3003\cdot 2^{-10}w^{-17} - \cdots, \end{aligned} \tag{1.3a}$$

$$\begin{aligned} z &= w \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{-2k-1}{k} (-3^{-1}w^2)^k = w \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{3k}{k} (3^{-1}w^2)^k \\ &= w + 3^{-1}w^3 + 3^{-1}w^5 + 4\cdot 3^{-2}w^7 + 55\cdot 3^{-4}w^9 + 91\cdot 3^{-4}w^{11} + \cdots. \end{aligned} \tag{1.3b}$$

Series (1.3b) converges for the value $w = 3^{-1}$ from (1.2b) to the "small" root:

$$z_1 = 3^{-1} + 3^{-4} + 3^{-6} + 4\cdot 3^{-9} + 55\cdot 3^{-13} + 91\cdot 3^{-15} + \cdots = 0.347296\cdots, \tag{1.4b}$$

whereas (1.3a) converges for each of the two values $w = \pm 3^{1/2}$ from (1.2a), and yields the numerically larger roots z_2 and z_3 of the given equation (1.1):

$$\begin{aligned} z_2, z_3 &= \pm 3^{1/2}(1 - 6^{-2}/2 - 105\cdot 6^{-6}/2 - 3003\cdot 6^{-9}/2 \\ &\qquad - 415701\cdot 6^{-12}/8 - \cdots) - z_1/2 \\ &= \pm 1.705737\cdots - 0.173648\cdots = 1.532089 \text{ or } -1.879385. \end{aligned} \tag{1.4a}$$

The inversion of the functions $w = f(z)$ in (1.2a) and (1.2b), by using the series (1.3a) and (1.3b) respectively, is a special case of a much more general theorem on the inversion of power series. The theorem is not very well known and, as stated here, it is new in form and generality. References to similar results will be given later. We shall prove the theorem in Section 2 by a simple and direct method involving contour integration, and we shall specialize it in Section 3 to obtain convergent series for all the roots of the general trinomial equation.

There are many other applications of the inversion formula including, for

example, finding the roots of Bessel functions, and deriving Stirling's formula with remainder for the Gamma function.

THEOREM 1. *For any nonzero integers p, q, s, let $w=f(z)$ and $z=g(w)$ be inverse functions defined for z^s near 0 (that is, for z near 0 when $s>0$ and for $1/z$ near 0 when $s<0$) by the convergent power series*

$$w^p = z^p[1 + \alpha(z^s)], \quad \text{where} \quad \alpha(z^s) = \sum_{k=1}^{\infty} a_k z^{ks}, \tag{1.5}$$

$$z^q = w^q[1 + \beta(w^s)], \quad \text{where} \quad \beta(w^s) = \sum_{k=1}^{\infty} b_k w^{ks}, \tag{1.6}$$

and assume that $w/z=(1+\alpha)^{1/p}$ is that pth root of $1+\alpha$ that is near 1 when $\alpha=\alpha(z^s)$ is small.

Then the coefficients b_k in the inverted power series (1.6) *are given explicitly as polynomials in the coefficients a_k of* (1.5) *by the inversion formula*

$$b_k = \frac{q}{ks+q} \sum_{r=1}^{k} \binom{-(ks+q)/p}{r} a_k^{(r)}, \tag{1.7}$$

where the $a_k^{(r)}$ are homogeneous polynomials of degree r in $a_1, \cdots, a_k$, defined implicitly by

$$[\alpha(z^s)]^r = \sum_{k=1}^{\infty} a_k^{(r)} z^{ks} \qquad (r = 1, 2, \cdots), \tag{1.8}$$

and defined explicitly as the sum of all ordered products of r factors in which the sum of subscripts is k.

For example, for $k=6$, $r=3$ the coefficient $a_k^{(r)}$ is

$$a_6^{(3)} = 3a_1^2a_4 + 6a_1a_2a_3 + a_2^3. \tag{1.9}$$

Inversion formulas essentially equivalent to (1.7) have been given by Polya-Szegö [6] and Kamber [3] with proofs based on the Bürmann-Lagrange formula

$$F(z) = F(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} F'(x)\phi^n(x) \right]_{x=0}, \qquad \text{if } w = z/\phi(z). \tag{1.10}$$

Kantz [4] recently gave a somewhat lengthy derivation (for $p=q=s=1$) based on determining successive derivatives of z with respect to w in terms of derivatives of w with respect to z. Both Kamber's determinant formulas and Kantz's formulas with partitions and factorials seem unnecessarily complicated for easy use.

2. Proof of the inversion formula. Choose any circle C, center at the origin in the z-plane, whose image in the w-plane under the mapping (1.5) is a simple closed curve C' that winds once around the origin in the w-plane. For positive s such a C can be found with a radius chosen small enough so that both $|\alpha(z^s)|$

<1 in and on C and $|\beta(w^s)|<1$ in and on C'. For negative s, a circle C of sufficiently large radius can be chosen so that $|\alpha(z^s)|<1$ outside and on C, and $|\beta(w^s)|<1$ outside and on C'.

Upon integration of z^{k-1} around C for $k\neq 0$, the integrated function z^k/k returns to its original value after a circuit, and the integral vanishes. However, for $k=0$ the integrated function $\ln z=\ln r+i\theta$ increases by $2\pi i$. Similar results hold when integrating w^{k-1} around C' in the w-plane. This proves an important and well-known result for contour integration, namely:

$$(2.1)\qquad \frac{1}{2\pi i}\int_C z^{k-1}dz=\begin{cases}1 \text{ if } k=0,\\ 0 \text{ if } k\neq 0;\end{cases}\qquad \frac{1}{2\pi i}\int_{C'} w^{k-1}dw=\begin{cases}1 \text{ if } k=0,\\ 0 \text{ if } k\neq 0.\end{cases}$$

Using (2.1) we can extract the coefficient b_k from (1.6) by dividing $1+\beta=(z/w)^q$ by w^{ks+1} and integrating the resulting series term by term:

$$(2.2)\qquad \frac{1}{2\pi i}\int_{C'}(z/w)^q w^{-ks-1}dw=b_k.$$

We move from the curve C' in the w-plane to the curve C in the z-plane by changing the variable of integration from w to z. Then we perform an integration by parts in which the integrated term $z^q w^{-ks-q}/2\pi i(ks+q)$ drops out when evaluated around C.

$$(2.3)\qquad b_k=\frac{1}{2\pi i}\int_C z^q w^{-ks-q-1}(dw/dz)dz=0+\frac{1}{2\pi i}\int_C \frac{w^{-ks-q}}{ks+q}qz^{q-1}dz.$$

Next we replace w by its value $z(1+\alpha)^{1/p}$ from (1.5) and use the binomial expansion:

$$(2.4)\qquad \begin{aligned} b_k&=\frac{q}{ks+q}\,\frac{1}{2\pi i}\int_C[1+\alpha(z^s)]^{-(ks+q)/p}z^{-ks-1}dz\\ &=\frac{q}{ks+q}\sum_{r=0}^{\infty}\binom{-(ks+q)/p}{r}\left\{\frac{1}{2\pi i}\int_C[\alpha(z^s)]^r z^{-ks-1}dz\right\}.\end{aligned}$$

Finally the expression in braces is seen to be $a_k^{(r)}$, when the defining equation (1.8) is substituted and (2.1) is applied. This $a_k^{(r)}$ vanishes for $r=0$ and for $r>k$, so the summation over r can be restricted to values from 1 to k. This completes the proof of the inversion formula (1.7).

3. Solution of the trinomial equation. Equation (1.1) is a special case of the general trinomial equation.

$$(3.1)\qquad z^{\lambda+\mu}+Az^{\mu}+AB=0 \qquad (\lambda,\ \mu \text{ positive integers, } AB\neq 0).$$

For purposes of symmetry it will be convenient to define C and ν by the equations

$$ABC = 1, \qquad \lambda + \mu + \nu = 0, \tag{3.2}$$

and to define a, b, c as selected roots of the equations

$$a^\lambda = -A, \qquad b^\mu = -B, \qquad c^\nu = -C. \tag{3.3}$$

Then instead of equations (1.2a) and (1.2b) we use the following three equivalent ways of writing (3.1) in forms like (1.5):

$$w^\lambda = a^\lambda = z^\lambda(1 + z^\nu/C) = -A, \tag{3.4a}$$

$$w^\mu = b^\mu = z^\mu(1 + z^\lambda/A) = -B, \tag{3.4b}$$

$$w^\nu = c^\nu = z^\nu(1 + z^\mu/B) = -C. \tag{3.4c}$$

Each of these three equations is a special case of (1.5) in which all coefficients a_k except a_1 vanish, and hence the $a_k^{(r)}$ all vanish except $a_k^{(k)} = a_1^k$. Thus the b_k in (1.7) each reduce to the single term for $r = k$, and we find from (1.6) the following inverted series:

$$z^q = \sum_{k=0}^{\infty} \frac{q}{k\nu + q}\binom{-(k\nu + q)/\lambda}{k} a^{k\nu+q}C^{-k} \qquad \text{for } |R| \leq \rho, \tag{3.5a}$$

$$z^q = \sum_{k=0}^{\infty} \frac{q}{k\lambda + q}\binom{-(k\lambda + q)/\mu}{k} b^{k\lambda+q}A^{-k} \qquad \text{for } |R| \leq \rho, \tag{3.5b}$$

$$z^q = \sum_{k=0}^{\infty} \frac{q}{k\mu + q}\binom{-(k\mu + q)/\nu}{k} c^{k\mu+q}B^{-k} \qquad \text{for } |R| \geq \rho, \tag{3.5c}$$

where the quantities R and ρ that appear in the convergence conditions are given by

$$\rho = \lambda^\lambda\mu^\mu(-\nu)^\nu > 0, \qquad (-1)^\nu R = A^\nu C^{-\lambda} = B^\lambda A^{-\mu} = C^\mu B^{-\nu}. \tag{3.6}$$

Either $|R/\rho| \leq 1$, and we find (below) by the nth root test that the two series (3.5a) and (3.5b) both converge and yield respectively λ and μ roots of (3.1) corresponding to different choices of a and b in (3.3); or $|R/\rho| > 1$, and series (3.5c) converges for each of $\lambda+\mu$ values of c defined by (3.3). Thus in every case all the roots of the general trinomial equation (3.1) are given by convergent series.

In applying the nth root convergence test to the series (3.5a), (3.5b), (3.5c) it is convenient to use a lemma that may be proved by using Stirling's formula for the Gamma function.

(3.7) LEMMA. $$\lim_{n\to\infty}\binom{nx + y}{n}^{1/n} = \frac{x^x}{(x-1)^{x-1}}.$$

Applying this to (3.5a) with $k = n$, we find that the limit of the absolute value of the nth root of the nth term is L_a, where

$$L_a = \left|(-\nu/\lambda)^{-\nu/\lambda}(\mu/\lambda)^{-\mu/\lambda}a^\nu C^{-1}\right| = \left|(-\nu)^{-\nu}\lambda^{-\lambda}\mu^{-\mu}(-A)^\nu C^{-\lambda}\right|^{1/\lambda} = |R/\rho|^{1/\lambda}. \tag{3.8a}$$

Corresponding test limits for (3.5b) and (3.5c) are $|R/\rho|^{1/\mu}$ and $|R/\rho|^{1/\nu}$, and these limits are less than 1 respectively when $|R/\rho|<1$ and when $|R/\rho|>1$. For the case $|R/\rho|=1$ it can be shown that all three series (3.5a, b, c) converge, but the convergence is slow and the series are not very useful for computation in this case.

The use of the two series (3.5a) and (3.5b) has already been illustrated in the series (1.3a) and (1.3b) of Section 1. Let us now illustrate the other case, when series (3.5c) converges to give all the roots, real and complex, of a trinomial equation such as

$$z^5 + 10z^2 - 20 = 0. \tag{3.9}$$

Here the values of the exponents and coefficients are as follows:

$$\lambda = 3,\quad \mu = 2,\quad \nu = -5,\quad \rho = 3^3 2^2 5^{-5} = 108/3125, \tag{3.10}$$

$$a^3 = -A = -10,\ b^2 = -B = 2,\ c^{-5} = -C = 1/20,\ R = (-A)^{-5}C^{-3} = 2/25 > \rho.$$

In the solution of (3.9) using the series (3.5c), the letter c stands for any one of the five fifth roots of 20, of which one is 1.82056. We may first expand z in powers of c, and then replace c^5 by 20 wherever it occurs. Thus the five roots of (3.9) are given by

$$\begin{aligned} z &= \sum_{k=0}^{\infty} \frac{1}{2k+1}\binom{(2k+1)/5}{k} c^{2k+1}(-2)^{-k} \\ &= c - c^3/10 + 0 + c^7/10^3 + c^9/10^4 \\ &\quad - 18c^{11}/10^6 - 56c^{13}/10^7 + 0 + 234c^{17}/10^9 + 308c^{19}/10^{10} \\ &\quad - 6688c^{21}/10^{12} - 23868c^{23}/10^{13} + 0 + \cdots, \end{aligned} \tag{3.11}$$

$$z_k = 0.9916c - 0.1027c^3 + 0.0222c^2 + 0.0023c^4, \text{ where } c = 1.82056e^{2\pi ki/5},$$

$$= 1.8053\epsilon - 0.6196\epsilon^3 + 0.0729\epsilon^2 + 0.0253\epsilon^4, \text{ where } \epsilon = e^{2\pi ki/5}.$$

Taking $\epsilon = 1$, we obtain the real root 1.2846. The imaginary roots are found to be $1.0072 \pm 2.1003i$ and $-1.6495 \pm .3855i$.

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THREE HYPERBOLAS ASSOCIATED WITH A TRIANGLE

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1. Preliminaries. a. If a conic (q) is circumscribed about a triangle (T), the triangle (T') formed by the tangents to (q) at the vertices of (T) may be called the *tangential triangle of* (T) *for* (q).

b. THEOREM. *A triangle* (T) *and its tangential triangle* (T') *for a conic* (q) *are in perspective; the center of perspectivity* M *and the axis of perspectivity* m *are pole and polar with respect to* (q) [1].

Moreover, *the point* M *and the line* m *are trilinear pole and polar for each of the two triangles* (T) *and* (T'), as is readily verified.

The point M and the line m will be referred to as the *Lemoine point* and the *Lemoine axis* of (T) for the conic (q).

c. *Note.* Given the triangle (T), and one of the three elements, M, m, (q), the remaining two are determined. This is an immediate consequence of the preceding article.

d. THEOREM. *Given a triangle* (T), *the locus of the trilinear poles, for* (T), *of the lines passing through a fixed point* M *is a conic* (q) *circumscribed about* (T) [2].

The given point M is the Lemoine point of (T) for (q).

e. Conversely, *a conic* (q) *circumscribed about a given triangle* (T) *is the locus of the trilinear poles, for* (T), *of the lines passing through a fixed point, namely, the Lemoine point of* (T) *for* (q).

2. The isotomic transformation.

a. THEOREM. *If a point describes a straight line* u, *its isotomic, conjugate point, for a given triangle* (T), *describes a conic* (q) *circumscribed about* (T) [3, p. 125].

Moreover, *the Lemoine axis of* (T) *for* (q) *is the isotomic (or reciprocal) transversal* u' *of* u *for* (T).

Conic (q) may be called the *isotomic conic* of line u for (T).

b. Conversely, *a conic* (q) *circumscribed about a triangle* (T) *is the isotomic conic of a straight line, namely, of the isotomic conjugate* u, *for* (T), *of the Lemoine axis* u' *of* (T) *for* (q).

c. The two preceding propositions (Sections 2a, 2b) gain in interest when the line u coincides with its own isotomic conjugate transversal. There are four such lines in the plane of a triangle (T), namely, the three sides of the medial triangle $(T')=A'B'C'$ of (T), and the line at infinity of the plane of (T). The trilinear poles, for (T), of these four lines are the respective vertices of the anticomplementary, or more briefly, *an-ry*, triangle $(T'')=A''B''C''$ of (T), and the common centroid G of the three triangles (T), (T'), and (T'').

Of these four lines, the line at infinity i seems to be the only one whose isotomic conic with respect to (T) has been considered [6].

If the Lemoine axis of a triangle (T) for a conic (E) circumscribed about (T) coincides with the line at infinity i, the tangential triangle of (T) for (E) coincides with the an-ry triangle (T'') of (T). Hence, the isotomic conic (E) of i for (T) coincides with the circumscribed minimal (or Steiner) ellipse (E) of (T) [5].

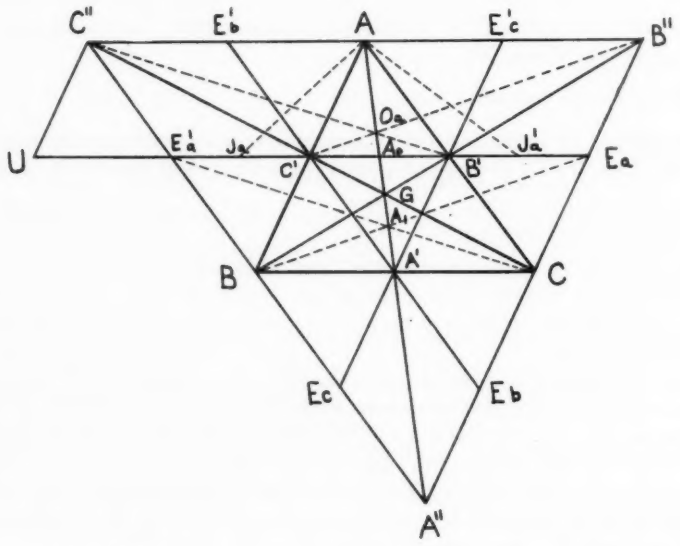
d. Moreover, the Lemoine point of (T) for (E) is the trilinear pole, for (T), of the Lemoine axis i of (T) for (E) (Section 1b), that is, the centroid G of (T). We have thus the following property of (E) (Section 1d): the trilinear poles, for a triangle (T), of the lines passing through the centroid of (T) lie on the circumscribed minimal ellipse of (T).

Observe that the Lemoine point G of (T) for (E) (Section 1b) is also the pole of the Lemoine axis i for the conic (E) (Section 1d). Hence, the point G is the center of the ellipse (E).

In what follows we shall consider the isotomic conics, for (T), of the other three self-conjugate isotomic transversals of (T).

3. A circumscribed hyperbola. a. The isotomic conic (H_a), for (T), of the self-conjugate isotomic transversal $B'C'$ which joins the midpoints B', C' of the sides AC, AB of (T) has the following two properties (Section 2):

(1) *The conic (H_a) is the locus of the isotomic conjugate points, for (T), of the points of $B'C'$.* (2) *(H_a) is the locus of the trilinear poles, for (T), of the lines passing through the vertex A'' of the an-ry triangle (T'') of (T).*



b. The conic (H_a) is circumscribed about (T) and the line $B'C'$ is the

Lemoine axis of (T) for (H_a) (Sections 3a, 2c), that is, the tangents to (H_a) at the vertices of (T) pass through the traces of $B'C'$ on the respectively opposite sides of (T). Hence, the conic (H_a) is tangent to the medians BB', CC', of (T), passing through the vertices B, C, while the tangent to (H_a) at A is the side $B''C''$ of the an-ry triangle (T'') of (T).

We shall refer to the point A as the *principal* point of the conic (H_a).

c. The points B', C' obviously lie inside the circumscribed Steiner ellipse (E) of (T) (Sections 2c, 2d). Hence line $B'C'$ meets (E) in two real points J_a, J_a'. The isotomic point I_a of J_a for (T) lies on the conic (H_a) (Section 3a) and on the line at infinity (Section 2c). The same is true for the isotomic I_a' of J_a', and it follows that (H_a) has two real points at infinity. Hence, the isotomic conic (H_a) for (T) of the self-conjugate isotomic transversal $B'C'$ is a hyperbola.

d. The median AGA' of (T) passes through the center G of the ellipse (E) (Section 2d). Now the chord J_aJ_a' is parallel to the tangent $B''C''$ to (E) at A. Hence the midpoint of J_aJ_a' is the trace A_0, on J_aJ_a', of the median AGA', which point is also the midpoint of the segment $B'C'$. Consequently, lines AJ_a, AJ_a' will meet the parallel BC to $B'C'$ in two points symmetrical with respect to A', that is, lines AJ_a, AJ_a' are a pair of isotomic lines for the angle A of (T). Therefore line AJ_a' passes through the isotomic conjugate point I_a (Section 3c) of the point J_a for the triangle (T). Similarly, line AJ_a passes through the isotomic point I_a' of J_a' for (T). Consequently (Section 3c), the two lines AJ_a, AJ_a' are a pair of asymptotic directions of the hyperbola (H_a).

Note. The points J_a, J_a' may be located, without drawing the ellipse (E), as follows. The tangents to (E) at points A, B meet in point C'' (Section 2c). Hence the polar, for (E), of the midpoint C' of AB passes through C'' and through the point at infinity of AB, that is, the polar of C' for (E) is the parallel to AB through C''. The trace U of that polar on line $B'C'$ is the homologous point of C' in the involution (I) of conjugate points, for ellipse (E), on line $B'C'$. The center of (I) is the point A_0, and points J_a, J_a' are the double elements of (I). Hence $(A_0J_a)^2=(A_0C')(A_0U)$. On the other hand, $C'U=AC''=BC=4(A_0C')$. Therefore, $A_0U=5(A_0C')$ and $AJ_a=AJ_a'=\sqrt{5}(A_0C')$.

e. Let $E_a=(B'C', A''B'')$, $E_a'=(B'C', A''C'')$. Point E_a is the midpoint of CB'', and AB is parallel to $A''B''$. Hence the pencil $B(CE_aB''A)$ is harmonic, and its section $(A'A_1GA)$ by AA'' is likewise harmonic. Thus the point $A_1=(BE_a, AG)$ is the harmonic conjugate of A with respect to the pair of points G, A'.

Now the tangents BB', CC' to (H_a) at points B, C (Section 3b) intersect in point G. Hence BC is the polar of G for (H_a) and, since point A lies on (H_a), point A_1 lies on the hyperbola (H_a).

Observe that point A_1 is the point of intersection of the diagonals BE_a, CE_a' of trapezoid BCE_aE_a'.

f. Point C is the midpoint of $A''B''$ and line AB is parallel to $A''B''$, whence $C'(A''CB''A)$ is a harmonic pencil. It follows that the range $(A''A_0O_aA)$, which

this pencil determines on the transversal AA'', is also harmonic. Thus the point $O_a=(C'B'', AGA'')$ is the harmonic conjugate of A'' for the points A, G.

Now the polar BC, for (H_a), of point G (Section 3e) is parallel to the tangent $B''C''$ of (H_a) at point A (Section 3b). Hence AGA'' is a diametral line of (H_a). On the other hand, lines BA_1E_a, $C'O_aB''$ are parallel, and C' is the midpoint of AB. Therefore, in triangle ABA_1, point O_a is the midpoint of side AA_1. Hence O_a is the center of (H_a) (Section 3e). Consequently,

(1) *The center of the hyperbola (H_a) is the harmonic conjugate, with respect to the principal point of (H_a) and the centroid of (T), of the vertex of (T'') which corresponds to the principal point of (H_a).*

(2) *Point A_1* (Section 3e) *is the diametric opposite of the principal point of (H_a) on this conic.*

Observe that the center O_a of (H_a) is the point of intersection of the diagonals $C'B''$, $B'C''$ of trapezoid $B'B''C''C'$.

g. We have (Section 3f), $-1=(A''GO_aA)=(AO_aGA'')$. If the latter harmonic division is compared with the harmonic division $(A'A_1GA)$ (Section 3e), it is seen that the three pairs of points A', A; G, G; A, A'' are corresponding elements in the homothecy $(G, -1:2)$ which transforms triangle (T) into its an-ry triangle (T''). Now the biratio (that is, the anharmonic ratio) is invariant under a homothecy, hence the fourth pair of points A_1, O_a are also homologous elements under the homothecy. Consequently, the diametric opposite of the principal point of the hyperbola (H_a) and the center of this conic are corresponding points in the homothecy $(G, -1:2)$.

A numerical verification of this proposition may be readily obtained. In triangle (T) we have $AG:AA'=2:3$, and hence $A_1G:A_1A'=2:3$ (Section 3e). Thus, if we put $A_1G=2$, we have successively,

$$A_1A'=3, \quad GA'=2+3=5, \quad AG=(2)(5)=10,$$
$$AA_1=10+2=12, \quad O_aA_1=12:2=6, \quad GO_a=6-2=4,$$
$$GA_1:GO_a=2:4=1:2.$$

That this last ratio is negative is evident in the figure.

h. (1) *The lines drawn through the center O_a of (H_a)* (Section 3f) *parallel to the lines AJ_a, AJ_a'* (Section 3d) *are the asymptotes of the hyperbola. They meet side BC of (T) in two isotomic points.*

Having located the asymptotes, the axes of (H_a) are obtained by drawing the bisectors of the angles formed by them.

(2) *Let R_a be the fourth common point, besides the vertices of (T), of (H_a) and the circumcircle (O) of (T). This point is the analog, for (H_a), of the Steiner point R of the Steiner ellipse (E)* [4]. *The isotomic of R_a is the point of intersection of the line $B'C'$ with the Longchamps axis of (T).*

(3) *The trilinear polar of R_a, for (T), joins the Lemoine point of (T) to the vertex A'' of (T'').*

4. Two other hyperbolas. The hyperbola (H_a) has been obtained as the

isotomic conic of the self-conjugate isotomic transversal $B'C'$ for (T) (Section 3a).

Considering the two remaining, analogous transversals $C'A'$, $A'B'$, two analogous hyperbolas (H_b), (H_c) are obtained. These conics are circumscribed about (T), are tangent at the vertices B, C, respectively, to the sides $C''A''$, $A''B''$ of the an-ry triangle (T'') of (T), and tangent to the medians of (T) at the remaining pairs of vertices of (T).

The centers, asymptotes, *etc.*, of these conics are obtained in the same way as the corresponding elements were obtained for the hyperbola (H_a). The explicit formulation of the corresponding propositions and their proofs offer no new difficulties and are omitted.

In what follows, some relations of the three conics to one another and to the circumscribed Steiner ellipse (E) of (T) will be considered.

5. Three circumscribed hyperbolas. a. The two hyperbolas (H_b), (H_c) pass through the vertex A of (T) and are tangent at that point to the median AA' of (T); hence the two conics have no point in common, besides the vertices of (T), and this is true also for the pairs of hyperbolas (H_c), (H_a); (H_a), (H_b).

Both the hyperbola (H_a) and the Steiner ellipse (E) are circumscribed about (T) and touch side $B''C''$ of the an-ry triangle (T'') of (T) at point A. It follows that these two conics have no other point in common, and a similar statement holds for (E) and each of the two hyperbolas (H_b), (H_c). Consequently, each of the four conics (H_a), (H_b), (H_c), (E) is tangent to the remaining three, and no two of them have a point in common, except the vertices of the triangle (T).

Observe that if (T) is an equilateral triangle, its circumcircle (O) is tangent to the sides of the an-ry triangle (T'') of (T) at the vertices of (T), whence the Steiner ellipse (E) coincides with (O). The three hyperbolas (H_a), (H_b), (H_c) are congruent, and a rotation of the entire figure about an angle of 120°, using the center of (O) for center of rotation, will bring the figure into coincidence with itself. The esthetic qualities of the figure may be worthy of the skill of a draftsman.

b. The triangle $O_aO_bO_c$ formed by the centers O_a, O_b, O_c of the hyperbolas (H_a), (H_b), (H_c) will be referred to as the *central triangle* of the three conics and will be denoted by ω.

The center O_a lies on the segment GA (Section 3f), and we have $GO_a:GA = 2:5$ (Section 3g). Similarly, $GO_b:GB = GO_c:GC = 2:5$. Therefore, the central triangle of the three hyperbolas (H_a), (H_b), (H_c) is homothetic to the basic triangle (T).

The center of the homothecy is the centroid G of (T); hence the point G is also the centroid of ω.

The point G is also the center of the ellipse (E) (Section 2d). Hence the above proposition may be stated as follows:

The center of the Steiner ellipse is the centroid of the central triangle of the three hyperbolas considered.

c. Let $(T_1)=A_1B_1C_1$ denote the triangle formed by the point A_1 (Section 3f) and its analogs B_1, C_1 for the hyperbolas (H_b), (H_c). By a method similar to the one used in connection with triangle ω (Section 5b), we may obtain the following results:

Triangle (T_1) *is homothetic to the given triangle* (T); *the center of homothecy is the centroid* G *of* (T), *and the ratio of homothecy is* $-1:5$ (Section 3g); *the centroid* G *of* (T) *is also the centroid of* (T_1).

d. Point G is the centroid of both triangle ω and triangle (T_1). Moreover, the points A_1, O_a, and, by analogy, the corresponding pairs of points B_1, O_b; C_1, O_c for (H_b), (H_c), are homologous points in the homothecy $(G, -1:2)$ (Section 3f), whence we conclude: The diametric opposites, on the respective conics, of the principal points of the three hyperbolas (H_a), (H_b), (H_c), are the vertices of the medial triangle of the triangle formed by the centers of those three curves.

e. The preceding proposition (Section 5d) gives rise to a considerable number of corollaries. Here is one example.

The line O_bO_c is parallel to BC (Section 5b) and therefore to $B''C''$; the latter line touches (H_a) in the diametric opposite A of the point A_1 common to O_bO_c and (H_a), whence the line O_bO_c is tangent to the hyperbola (H_a).

Analogous considerations apply to the lines O_cO_a, O_aO_b. Consequently, the line of centers of any two of the three hyperbolas considered is tangent to the third hyperbola, the point of contact being the diametric opposite of the principal point of the latter curve.

f. We have (Section 3g) $GO_a:GA=GA_1:GA'=2:5$, and similarly, for the analogous pairs of points O_b, B; B_1, B' and O_c, C; C_1, C' relative to (H_b) and (H_c). Hence the figure formed by triangles ω and (T_1) corresponds to the figure formed by (T) and its medial triangle $(T')=A'B'C'$ in the homothecy $(G, 2:5)$. Thus, if we construct for triangle ω the conics analogous to the conics (H_a), (H_b), (H_c), (E), of (T), the former set of conics will correspond to the latter set in the homothecy $(G, 2:5)$. In the conics so obtained we may consider the centers of the conics, *etc.*

The same construction may be applied again to the new figure, and so on, *ad infinitum*.

The same thing may be done in the opposite direction. We may construct the figure which corresponds to the figure (T), (T'), and its conics, in the homothecy $(G, 5:2)$. The new hyperbolas will have for their centers the points $A, B, C, \cdots$. This process, too, may be repeated *ad infinitum*.

g. Let R_b, R_c be the fourth points of intersection of (H_b), (H_c) with the circumcircle (O) of (T). We have (Sections 3h (2) and 3h (3)):

The four points R_a, R_b, R_c, R *have for their isotomic points, for* (T), *the traces of the Longchamps axis of* (T) *on the sides of the medial triangle* (T') *of* (T), *and the point at infinity of that axis; the trilinear polars, for* (T), *of the same four points are the lines which join the Lemoine point of* (T) *to the vertices of triangle* (T'') *and to the common centroid* G *of* (T) *and* (T'').

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"IF THIS BE TREASON . . . "

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If I had to name one trait that more than any other is characteristic of professional mathematicians, I should say that it is their willingness, even eagerness, to admit that they are wrong. A sure way to make an impression on the mathematical community is to come forward and declare, "You are doing such-and-such all wrong and you should do it *this* way." Then everybody says, "Yes, how clever you are", and adopts your method. This of course is the way progress is made, but it leads to some curious results. Once upon a time square roots of numbers were found by successive approximations because nobody knew of a better way. Then somebody invented a systematic process and everybody learned it in school. More recently it was realized that very few people ever want to extract square roots of numbers, and besides the traditional process is not really very convenient. So now we are told to teach root extraction, if we teach it at all, by successive approximations. Once upon a time people solved systems of linear equations by elimination. Then somebody invented determinants and Cramer's rule and everybody learned that. Now determinants are regarded as old-fashioned and cumbersome, and it is considered better to solve systems of linear equations by elimination.

We are constantly being told that large parts of the conventional curriculum are both useless and out of date and so might better not be taught. Why teach computation by logarithms when everybody who has to compute uses at least a desk calculator? Why teach the law of tangents when almost nobody ever wants to solve an oblique triangle, and if he does there are more efficient ways? Why teach the conventional theory of equations, and especially why illustrate it with ill-chosen examples that can be handled more efficiently by other methods? As a professional mathematician, I am a sucker for arguments like these. Yet, sometimes I wonder.

There are a few indications that there is a reason for the survival of the traditional curriculum besides the fact that it is traditional. When I was teaching mathematics to future naval officers during the war, I was told that the Navy had found that men who had studied calculus made better line officers than men who had not studied calculus. Nothing is clearer (it was clear even to the Navy) than that a line officer never has the slightest use for calculus. At the most, his duties may require him to look up some numbers in tables and do a little arithmetic with them, or possibly substitute them into formulas. What is the explanation of the paradox?

I think that the answer is supplied by a phenomenon that everybody who teaches mathematics has observed: the students always have to be taught what they should have learned in the preceding course. (We, the teachers, were of course exceptions; it is consequently hard for us to understand the deficiencies of our students.) The average student does not really learn to add fractions in arithmetic class; but by the time he has survived a course in algebra he can add numerical fractions. He does not learn algebra in the algebra course; he learns it in calculus, when he is forced to use it. He does not learn calculus in the calculus course, either; but if he goes on to differential equations he may have a pretty good grasp of elementary calculus when he gets through. And so on through the hierarchy of courses; the most advanced course, naturally, is learned only by teaching it.

This is not just because each previous teacher did such a rotten job. It is because there is not time for enough practice on each new topic; and even if there were, it would be insufferably dull. Anybody who has really learned to interpolate in trigonometric tables can also interpolate in air navigation tables, or in tables of Bessel functions. He should learn, because interpolation is useful. But one cannot drill students on mere interpolation; not enough, anyway. So the students solve oblique triangles in order (among other things) to practice interpolation. One must not admit this to the students, but one may as well realize the facts.

Consequently, I claim that there is a place, and a use, even for nonsense like the solution of quartics by radicals, or Horner's method, or involutes and evolutes, or whatever your particular candidates for oblivion may be. Here are problems that might conceivably have to be solved; perhaps the methods are not the most practical ones; but that is not the point. The point is that in solving the problems the student gets practice in using the necessary mathematical tools, and gets it by doing something that has more motivation than mere drill. This is not the way to train mathematicians, but it is an excellent way to train mathematical technicians. Now we can understand why calculus improves the line officer. He needs to practice very simple kinds of mathematics; he gets this practice in less distasteful form by studying more advanced mathematics.

It is the fashion to deprecate puzzle problems and artificial story problems. I think that there is a place for them too. Problems about mixing chemicals or

sharing work, however unrealistic, give good practice and even have a good deal of popular appeal: witness the frequency with which puzzle problems appear in newspapers, magazines, and the flyers that come with the telephone bill. There was once a story in *The Saturday Evening Post* whose plot turned on the interest aroused by a perfectly preposterous diophantine problem about sailors, coconuts, and a monkey. It is absurd to claim that only "real" applications should be used to illustrate mathematical principles. Most of the real applications are too difficult and/or involve too many side issues. One begins the study of French with simple artificial sentences, not with the philosophical writings of M. Sartre. Similarly one has to begin the study of a branch of mathematics with simple artificial problems.

We may dislike this state of affairs, but as long as it exists we must face it. It would be pleasant to teach only the new and exciting kinds of mathematics; it would be comforting to teach only the really useful kinds. The traditional topics are some of the topics that once were either new and exciting, or useful. They have persisted partly by mere inertia—and that is bad—but partly because they still serve a real purpose, even if it is not their ostensible purpose. Let us keep this in mind when we are revising the curriculum.

NON-LINEAR RECURRENCE RELATIONS FOR CERTAIN CLASSICAL FUNCTIONS

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Let $\{f_n(x)\}$ be a sequence of functions. The combinations $(f_n')^2-f_nf_n''$, $(f_n')^2-f_{n-1}'f_{n+1}'$, and $f_n^2-f_{n-1}f_{n+1}$ have appeared in recent papers. We consider some relationships involving these functions, when $\{f_n(x)\}$ is the sequence of ultraspherical, Hermite, and generalized Laguerre polynomials using Szegö's [1] notation and the Bessel functions of the first kind.

THEOREM 1. *If $f_n(x)$ is a polynomial of degree n, $f_0(x)=1$, $f_1(x)=2\lambda x$, and*

$$(1)\quad (1-x^2)[(f_n')^2-f_{n-1}'f_{n+1}'] = n(n+2\lambda)f_n^2-(n+1)(n+2\lambda-1)f_{n-1}f_{n+1}$$

*for each $n\geqq 1$, where λ is not of the form $-(n-1)/2$, $n=0, 1, \cdots$, then $f_n(x)=P_n^{(\lambda)}(x)=P_n(x)$ for $n=0, 1, \cdots$, where $\{P_n(x)\}$ is the sequence of ultraspherical polynomials.**

Proof. It may be verified directly that $f_2(x)=P_2(x)$. Assume that $f_m(x)=P_m(x)$ for $0\leqq m\leqq n$ where $n\geqq 2$. Then

* This theorem is a generalization of a result of Carlitz [2].

$$(2)\quad (1-x^2)[(P_n')^2 - P_{n-1}'f_{n+1}'] = n(n+2\lambda)P_n^2 - (n+1)(n+2\lambda-1)P_{n-1}f_{n+1}.$$

Let $f_{n+1}(x)=P_{n+1}(x)+g(x)$ where $g(x)$ is a polynomial of degree $\leqq n+1$. Substituting in (2) and using the fact [3] that $\{P_n(x)\}$ satisfies (1), we obtain

$$(3)\qquad (1-x^2)P_{n-1}'g' = (n+1)(n+2\lambda-1)P_{n-1}g.$$

By comparing the coefficients of x^{2n} in (3), one sees that the coefficient of x^{n+1} in $g(x)$ is uniquely determined and so $g(x)$ is of degree $\leqq n$. Since $(1-x^2)P_{n-1}'$ and P_{n-1} have no polynomial factor of degree $\geqq 1$, it follows from (3) that

$$g(x) = C(1-x^2)P_{n-1}', \qquad g'(x) = C(n+1)(n+2\lambda-1)P_{n-1},$$

where C is a constant. If $C\neq 0$, this gives

$$(1-x^2)P_{n-1}'' - 2xP_{n-1}' = (n+1)(n+2\lambda-1)P_{n-1},$$

which, together with the known [1] differential equation

$$(1-x^2)P_{n-1}'' - (2\lambda+1)xP_{n-1}' + (n-1)(n+2\lambda-1)P_{n-1} = 0,$$

gives a contradiction. Hence $C=0$, $f_{n+1}(x)=P_{n+1}(x)$ and the theorem is proved.

THEOREM 2. *If* $f_0(x)=1$, $f_1(x)=2\lambda x$, *and*

$$(4)\quad (1-x^2)^2[(f_n')^2 - f_nf_n''] = n(n+2\lambda)f_n^2 - (n+1)(n+2\lambda-1)f_{n-1}f_{n+1} - x(1-x^2)f_nf_n',$$

for each $n\geqq 1$, *where* λ *is not of the form* $-(n-1)/2$, $n=0, 1, \cdots$, *then* $f_n(x) = P_n^{(\lambda)}(x)=P_n(x)$ *for* $n=0, 1, \cdots$.

Proof. Since [4] the sequence $\{P_n(x)\}$ satisfies (4), and (4) determines f_{n+1} uniquely in terms of f_{n-1} and f_n, the theorem follows.

THEOREM 3. *If* $f_0(x)=1$, $f_1(x)=2\lambda x$, *and*

$$(1-x^2)^2[(f_n')^2 - f_nf_n''] = n(n+2\lambda+x^2)f_n^2 - (n+1)(n+2\lambda-1)f_{n-1}f_{n+1} - (n+2\lambda-1)xf_{n-1}f_n$$

for each $n\geqq 1$, *where* λ *is not of the form* $-(n-1)/2$, $n=0, 1, \cdots$, *then* $f_n(x) = P_n^{(\lambda)}(x)=P_n(x)$ *for* $n=0, 1, \cdots$.

Proof. Since [1] we have $(1-x^2)P_n' = -nxP_n+(n+2\lambda-1)P_{n-1}$, then, in view of Theorem 2,

$$(1-x^2)^2[(P_n')^2 - P_nP_n''] = n(n+2\lambda+x^2)P_n^2 - (n+1)(n+2\lambda-1)P_{n-1}P_{n+1} - (n+2\lambda-1)xP_{n-1}P_n.$$

The remainder of the proof is similar to that in Theorem 2.

THEOREM 4. *If* $f_0(x)=1$, $f_1(x)=2x$, *and*

$$(f_n')^2 - f_n f_n'' = 2n[f_n^2 - f_{n-1}f_{n+1}] \tag{5}$$

for each $n \geqq 1$, *then* $f_n(x)=H_n(x)$ *for* $n=0, 1, \cdots$, *where* $\{H_n(x)\}$ *is the sequence of Hermite polynomials.*

Proof. Since [5, 6] the sequence $\{H_n(x)\}$ satisfies (5), the proof is similar to that in Theorem 2.

THEOREM 5. *If* $f_0(x)=1$, $f_1(x)=-x+\alpha+1$, *and*

$$x^2[(f_n')^2 - f_n f_n''] = n(n+\alpha+1)f_n^2 - (n+1)(n+\alpha)f_{n-1}f_{n+1} \tag{6}$$

for each $n \geqq 1$, *where* α *is not a negative integer, then* $f_n(x)=L_n^{(\alpha)}(x)=L_n(x)$ *for* $n=0, 1, \cdots$, *where* $\{L_n(x)\}$ *is the sequence of generalized Laguerre polynomials.*

Proof. Since [4] the sequence $\{L_n(x)\}$ satisfies (6), the proof is similar to that in Theorem 2.

Note that equation (6) may be written in the form [4]

$$x^2[(f_n')^2 - f_n f_n''] + \alpha f_n^2 = (n+1)(n+\alpha)[f_n^2 - f_{n-1}f_{n+1}].$$

Since $(L_n')^2 - L_n L_n'' > 0$ by [6], it follows, as is known [3, 4], that

$$n(n+\alpha+1)L_n^2 - (n+1)(n+\alpha)L_{n-1}L_{n+1} \geqq 0,$$

$$L_n^2 - L_{n-1}L_{n+1} \geqq 0, \qquad \text{if } \alpha \geqq 0.$$

THEOREM 6. *If* $f_n(x)$ *is a polynomial of degree* n, $f_0(x)=1$, $f_1(x)=-x+\alpha+1$, *and*

$$x[(f_n')^2 - f_{n-1}'f_{n+1}'] = nf_n^2 - (n+1)f_{n-1}f_{n+1} + f_{n-1}f_n \tag{7}$$

for each $n \geqq 1$, *where* α *is a constant, then* $f_n(x)=L_n^{(\alpha)}(x)=L_n(x)$ *for* $n=0, 1, \cdots$.

Proof. We first use [1] the relations

$$xL_n' = nL_n - (n+\alpha)L_{n-1} = (n+1)L_{n+1} - (n+\alpha+1-x)L_n, \tag{8}$$

$$nL_n = (-x+2n+\alpha-1)L_{n-1} - (n+\alpha-1)L_{n-2}, \tag{9}$$

to show that $\{L_n(x)\}$ satisfies (7).

Let $E=x[(L_n')^2 - L_{n-1}'L_{n+1}']$. Then, by (8) and (9),

$$\begin{aligned} xE &= [nL_n - (n+\alpha)L_{n-1}]^2 \\ &\quad - [nL_n - (n+\alpha-x)L_{n-1}][(n+1)L_{n+1} - (n+\alpha+1)L_n] \\ &= n(2n+\alpha+1)L_n^2 - [(n+\alpha)(3n+\alpha+1) - (n+\alpha+1)x]L_{n-1}L_n \\ &\quad + (n+\alpha)^2 L_{n-1}^2 - n(n+1)L_nL_{n+1} + (n+1)(n+\alpha-x)L_{n-1}L_{n+1}. \end{aligned}$$

We replace $(n+\alpha)L_{n-1}$ by its value obtained from (9) after n is changed to $n+1$. This gives

$$(n+\alpha)E = [(n+1)(n+\alpha+1) - x]L_n^2 + (n+1)^2 L_{n+1}^2 + [n+1][x-(2n+\alpha+2)]L_nL_{n+1}.$$

In the terms with the variable coefficients, namely, $-xL_n^2+(n+1)xL_nL_{n+1}$, we replace the factor xL_n by its value obtained from (9) after n is replaced by $n+1$. We obtain $E=nL_n^2-(n+1)L_{n-1}L_{n+1}+L_{n-1}L_n$, which shows that $\{L_n(x)\}$ satisfies (7).

We are now ready to prove the theorem. It may be verified directly that $f_2=L_2$. Assume that $f_m(x)=L_m(x)$ for $0\leqq m\leqq n$ where $n\geqq 2$.

Then,

$$x[(L_n')^2 - L_{n-1}'f_{n+1}'] = nL_n^2 - (n+1)L_{n-1}f_{n+1} + L_{n-1}L_n. \tag{10}$$

Let $f_{n+1}(x)=L_{n+1}(x)+g(x)$, where $g(x)$ is a polynomial of degree $\leqq n+1$. Substituting in (10) and using the above result that $\{L_n(x)\}$ satisfies (7), we find that $xL_{n-1}'g'=(n+1)L_{n-1}g$; this shows that if Cx^k is the term in g which involves the highest power of x, then $C=0$. Hence $g(x)=0$ and $f_{n+1}=L_{n+1}$.

THEOREM 7. *If $f_0(x)=J_0(x)$, $f_1(x)=J_1(x)$, and*

$$x[(f_n')^2 = f_nf_n''] = x[f_n^2 - f_{n-1}f_{n+1}] + f_nf_n' \tag{11}$$

for each $n\geqq 1$, then $f_n(x)=J_n(x)$ for $n=0, 1, \cdots$, where $J_n(x)$ is the Bessel function of the first kind of order n.

Proof. Since [4] the sequence $\{J_n(x)\}$ satisfies (11), the proof follows as in Theorem 2.

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MATHEMATICAL NOTES

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Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

ELEMENTARY PROOFS OF THE COMMUTATIVITY OF p-RINGS

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1. Introduction. A p-ring, first introduced by McCoy and Montgomery [2], is a ring with unit* 1 satisfying $a^p=a$, $pa=0$ (p prime). The object of this note is to give some elementary proofs of the following

THEOREM 1. *A p-ring is commutative.*†

First proof. Since $x^p=x$ in a p-ring, we immediately obtain (compare Lemma 1 of [1]) $(x^{p-1}yx^{p-1}-x^{p-1}y)^2=(x^{p-1}yx^{p-1}-yx^{p-1})^2=0$, and this implies

$$x^{p-1}y = x^{p-1}yx^{p-1} = yx^{p-1}. \tag{1.1}$$

Since $x^p=x$ and x^{p-1} is in the center, it is apparent at once that

$$(xy - yx)x^{p-1} = x^{p-1}(xy - yx) = xy - yx. \tag{1.2}$$

Since this is true for every x and y, it remains true if x is replaced by $x+i$, where $i=1, \cdots, p-1$. Since, however, $xy-yx$ is unchanged under this replacement, it follows that

$$(x + i)^{p-1}(xy - yx) = xy - yx \qquad (i = 0, \cdots, p - 1). \tag{1.3}$$

Now, an elementary number-theoretic result states that, since p is prime,

$$(x + 1)(x + 2) \cdots (x + (p - 1)) = x^{p-1} + (p - 1). \tag{1.4}$$

Since $pa=0$, a combination of (1.3) and (1.4) gives

$$(x + 1)(x + 2) \cdots (x + (p - 1))(xy - yx) = 0. \tag{1.5}$$

Replacing, in (1.5), x by $x+1$ we obtain

$$\begin{aligned}(x + 2)(x + 3) \cdots (x + (p - 1))x(xy - yx)& \\ = x(x + 2)(x + 3) \cdots (x + (p - 1))(xy - yx) = 0.&\end{aligned} \tag{1.6}$$

Subtracting (1.6) from (1.5), we get

$$1\cdot(x + 2)(x + 3) \cdots (x + (p - 1))(xy - yx) = 0. \tag{1.7}$$

Similarly, upon replacing x by $x+1$ in (1.7) and subtracting as above, we obtain

* McCoy and Montgomery do not require the existence of a unit.

† Forsythe and McCoy [1] have already given an elementary proof of Theorem 1. In this note, however, the approach is quite different from [1] in being essentially of number-theoretic nature.

$$2!(x+3)(x+4)\cdots(x+(p-1))(xy-yx)=0. \tag{1.8}$$

Continuing this process, we ultimately obtain

$$(p-2)!(x+(p-1))(xy-yx)=0. \tag{1.9}$$

Now, replacing again x by $x+1$ in (1.9), we get

$$(p-2)!x(xy-yx)=0. \tag{1.10}$$

Subtracting the last two equations, and making use of $(p-1)!=p-1$ in a p-ring, we obtain $(p-1)(xy-yx)=0$. Hence $xy-yx=0$, $xy=yx$, and the theorem is proved.

Second proof. This proof uses some of the results of the first proof. Thus, using (1.3), (1.1) and (1.4), we obtain

$$\begin{aligned} xy-yx &= (xy-yx)^p \\ &= \{x^{p-1}(xy-yx)\}\{(x+1)^{p-1}(xy-yx)\}\cdots\{(x+(p-1))^{p-1}(xy-yx)\} \\ &= \{x(x+1)\cdots(x+(p-1))\}^{p-1}(xy-yx)^p=0. \end{aligned}$$

Hence $xy=yx$, and the proof is complete.

In conclusion, I wish to express my gratitude to the referee for his valuable suggestions.

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ON THE NOTE "ON THE PROPAGATION OF ERROR BY MULTIPLICATION" BY PERRY AND MORELOCK

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It may be worth pointing out that the note cited in the title [1] represents a familiar manipulation of bivariate probability distributions. If X and Y are independent random variables, uniformly distributed in the intervals $[a-1/2, a+1/2]$ and $[b-1/2, b+1/2]$ respectively, then the distribution of $W=XY-ab$ may readily be found. One example of a textbook discussion of this standard approach is Chapter 10 of [2].

Assume that $1/2 \leqq a < b$. This assumption, together with that of independence, is implicit in the note by Perry and Morelock. The independence assumption, in particular, needs careful examination for many applications.

The joint p.d.f. (probability density function) of X and Y is 1 for $a-1/2 \leqq x \leqq a+1/2$ and $b-1/2 \leqq y \leqq b+1/2$; and it is zero elsewhere. (I use lower case letters for the arguments of a p.d.f. relating to the random variables denoted by the corresponding capitals.) If we let $W=XY-ab$ and $Z=Y$ (to be wholly

explicit) then the Jacobian of the transformation is $1/z$ and the joint p.d.f. of W and Z is $1/z$ in the quadrilateral bounded by the lines $z=b\pm 1/2$, $z=(x+ab)/(a\pm 1/2)$, and zero elsewhere. Integrating out over z, one obtains as the p.d.f. of W:

$$\begin{aligned}
&\ln(w+ab)-\ln[(a-1/2)(b-1/2)] && \text{for } (-2a-2b+1)/4\leqq w\leqq(2a-2b-1)/4,\\
&\ln[(b+1/2)/(b-1/2)] && \text{for } (2a-2b-1)/4\leqq w\leqq(-2a+2b-1)/4,\\
&-\ln(w+ab)+\ln[(a+1/2)(b+1/2)] && \text{for } (-2a+2b-1)/4\leqq w\leqq(2a+2b+1)/4,\\
&0 && \text{elsewhere.}
\end{aligned}$$

And, from this, the final expressions given by Perry and Morelock may readily be found by integration.

The reader of [1] may be puzzled by the fact that the graph on page 179 of [1] is not a graph of the function $P\{E>K\}$, as suggested there, but rather a graph of the p.d.f. given in the preceding paragraph of this note. Perry and Morelock do not explicitly state the p.d.f.; their $P\{E>K\}$ is the complement of the usual cumulative distribution function. (Their E is my W.)

Finally, the graph of [1] is slightly misleading for two reasons. First, there is inordinate rounding of abscissas; the points labeled -6.8, -1.3, .7, and 7.2 could have been labeled -6.75, -1.25, .75, and 7.25, respectively and exactly. Second, the two slanted straight line segments of the graph are actually slightly curved, the left, concave down, and the right, concave up; these curvatures, although small, would just be noticeable on the scale of the exhibited figure.

The editor is informed that these concavities were present on the graph submitted with the note; apparently they became lost during preparations for printing.

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A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION

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1. Introduction. If $f(x)$ is a complex-valued, continuous function of the real variable x such that for every real x, y

$$f(x)f(y) = f(x+y) \tag{1}$$

then we know that either $f(x)=0$, or there exists a complex number α such that $f(x)$ can be written in the form $e^{\alpha x}$ for all real x. Actually, as is well known, the condition of continuity is too stringent and can be replaced by mere measurability.

However, in some practical applications, whereas continuity may be expected, the defining condition (1) may not be readily verifiable for every possible

pair x, y. The author was faced with such a situation and had the following problem:

$f(x)$ is a complex-valued, continuous function of the real variable x, having the property $\mathcal{P}$ that if, for any real x and any positive integer n, $k_n(x)$ denote the number of distinct points in the set $f(x)$, $\{f(x/2)\}^2$, $\{f(x/3)\}^3$, $\cdots$, $\{f(x/n)\}^n$, then $k_n(x) \leqq$ some finite k for all x and n.

What can be said about $f(x)$? Can it, for instance, always be written in the form $e^{\alpha x+\beta}$? The author has been unable to obtain the complete answer, but a partial solution is given in the next section.

2. Partial solution of the problem. We shall now restrict ourselves to real-valued functions, and prove the following

THEOREM. *Let $f(x)$ be a real-valued, continuous function of $x \geqq 0$ having the property $\mathcal{P}$ defined in Section 1. Further, let $f(x) > 0$ for some $x \geqq 0$. Then there exists a real number α such that, for all $x \geqq 0$, $f(x)$ can be written in the form $e^{\alpha x}$.*

Proof. Since $f(x) > 0$ for some x, therefore $f(x) > 0$ throughout some closed interval $[a, b]$. It will be assumed that $a > 0$. In this interval, the function $g(x) = [\log f(x)]/x$ is defined and continuous.

For any $x > ab/(b-a)$, there is at least one integer r such that $a \leqq x/r \leqq b$. Let $n(x)$ and $m(x)$ be respectively the smallest and largest such integers. Then for all n such that $n(x) \leqq n \leqq m(x)$, $g(x/n) = n\{\log f(x/n)\}/x$ is defined. From property $\mathcal{P}$ it follows that, however large x may be, there can be at most k distinct points in the set

$$g\{x/n(x)\}, \quad g\{x/[n(x)+1]\}, \cdots, g\{x/m(x)\}.$$

But we shall show that this implies the constancy of $g(x)$ throughout $[a, b]$.

Since $g(x)$ is continuous in $[a, b]$, therefore, unless it is a constant, there are $k+1$ points $a_0, a_1, \cdots, a_k$ in $[a, b]$ such that $g(a_r) \neq g(a_s)$ if $r \neq s$. It follows that there exists a set of non-overlapping intervals $I_0, I_1, \cdots, I_k$ such that $a_r \in I_r \subset [a, b]$, $r = 0, \cdots, k$, and $g(x_r) \neq g(x_s)$ if $r \neq s$, $x_r \in I_r$ and $x_s \in I_s$.

Now, it is easy to see that by choosing x large enough, we can find positive integers $n_0, n_1, \cdots, n_k$ such that $x/n_r \in I_r$. Hence, the set $g\{x/n(x)\}, \cdots, g\{x/m(x)\}$ contains at least $k+1$ distinct points, which was previously seen to be impossible, on account of property $\mathcal{P}$. Consequently, $g(x)$ equals a constant α in $[a, b]$, so that throughout this interval, $f(x) = e^{\alpha x}$.

Further, it is easy to see that $f(x) \neq 0$ for all $x > a$. For, otherwise there is a $c > a$ which is the g.l.b. of all zeros of $f(x)$ to the right of a, and by continuity $f(c) = 0$. On the other hand, if we consider any sequence of points c_n in (a, c) such that $c_n \to c$ as $n \to \infty$, then we know that $f(x) > 0$ in $[a, c_n]$, so that $f(c_n) = e^{\alpha c_n} \to e^{\alpha c}$ as $n \to \infty$. This is impossible. Hence, $f(x)$ has no zero to the right of a, and $f(x) = e^{\alpha x}$ for all $x \geqq a$. In a similar manner, it can be seen that $f(x)$ has no zero in $(0, a)$, and that $f(x) = e^{\alpha x}$ for all $x \geqq 0$.

COROLLARY. *If $f(x)$ is a real-valued, continuous function of the real variable x having the property $\mathcal{P}$, then one of the following alternatives must hold:*

(a) $f(x) \equiv 0$;

(b) $f(x) > 0$ *for some* x, *in which case* $f(x) > 0$ *for all* x, *and there exist real numbers* α, β *such that* $f(x)$ *can be written in the form* $e^{\alpha x}$ *for* $x \geqq 0$, *and* $e^{\beta x}$ *for* $x \leqq 0$;

(c) $f(x) < 0$ *for some* x, *in which case* $f(x) < 0$ *for all* x, *and there exist real numbers* α, β *such that* $f(x)$ *can be written in the form* $-e^{\alpha x}$ *for* $x \geqq 0$ *and* $-e^{\beta x}$ *for* $x \leqq 0$.

3. Unsolved problem. Coming back to the problem of a complex-valued, continuous $f(x)$ having the property $\mathcal{P}$, we can express $f(x)$ as the product of two functions $|f(x)|$ and $f(x)/|f(x)|$ having the property $\mathcal{P}$. Of these two, $|f(x)|$ is real-valued and can be taken care of by the previous section.

It therefore remains to consider a continuous function $f(x)$ of unit modulus and having property $\mathcal{P}$. It is conjectured that in this case, the real line can be broken up into intervals in each of which $f(x)$ can be written in the form $\mu e^{i\alpha x}$ where μ is a root of unity and α is a real number. The author has been unable to prove or disprove this conjecture.

REMARKS CONCERNING THE NON-EXISTENCE OF ODD PERFECT NUMBERS

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Let $\sigma(n)$ be the sum of the divisors of the positive integer n. Then n is called a perfect number if $\sigma(n) = 2n$. It has long been conjectured that there are no odd perfect numbers. Euler [2] proved that if n is an odd perfect number, then

$$n = p^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}, \tag{1}$$

where $p, q_1, q_2, \cdots, q_t$ are distinct primes and $p \equiv 1 \equiv \alpha \pmod 4$. Sylvester [6] proved that $t \geqq 4$, and, in fact, that $t \geqq 7$ if $n \not\equiv 0 \pmod 3$. In recent years further necessary conditions have been discovered for the odd integer n to be perfect. In this note we shall show that certain other conditions are necessary.

If the number n, given by (1), is to be perfect, we must have

$$\sigma(p^{\alpha}) \prod_{i=1}^{t} \sigma(q_i^{2\beta_i}) = 2p^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}.$$

To prove that n is not perfect it is therefore sufficient to show that $\sigma(p^{\alpha})$ or one of the $\sigma(q_i^{2\beta_i})$ has a prime factor which does not divide n. We shall assume throughout this note that $n \not\equiv 0 \pmod 3$.

If $n \equiv 2 \pmod 3$, then $p \equiv 2 \pmod 3$, and $\sigma(p^{\alpha})$, which is divisible by $p+1$, is divisible by 3. Hence, if n is to be perfect we must have $n \equiv 1 \pmod 3$. Now, if this is true, $\sigma(p^{\alpha}) \equiv \alpha + 1 \pmod 3$, and therefore, if $\alpha + 1$ is divisible by 3, n is not perfect.

This can be generalized as follows. Let r be any prime not dividing n, and let p belong to the exponent e modulo r. Then, n is not perfect if $\alpha + 1 \equiv 0 \pmod{er}$. For, in this case, $\sigma(p^{\alpha})$ is divisible by $p^{e(r-1)} + p^{e(r-2)} + \cdots + p^{e} + 1$, which is divisible by r. As an application of this remark, we see that n is not perfect if n is not divisible by 5, $\alpha + 1$ is divisible by 5, and $p \equiv 1$ or $4 \pmod 5$.

It was proved by Steuerwald [5] that n, given by (1), is not perfect if $\beta_1=\beta_2=\cdots=\beta_t=1$. We shall extend this by showing that n is not perfect if $\beta_i\equiv 1 \pmod 3$ for $i=1,\cdots,t$. Assume that n is perfect. If, for any i, $q_i \equiv 1 \pmod 3$, $\sigma(q_i^{2\beta_i})$, and therefore n, is divisible by 3. Hence we may assume that $q_i\equiv 2 \pmod 3$ for $i=1,\cdots,t$. Suppose that we have shown that n is not divisible by any prime less than the prime q, and that n is divisible by q. If $q=p$, n is divisible by $(q+1)/2$, which contains a prime factor less than q. Hence, q is one of the q_i, and n is divisible by $\sigma(q^2)=q'$. If q' is composite, it contains a prime factor less than q. Hence, q' is a prime. Since $q'\equiv 1 \pmod 3$, we must have $q'=p$. But then n is divisible by $(q'+1)/2=\frac{1}{2}(q^2+q)+1=q''$. If q'' is composite, it contains a prime factor less than q. Hence, q'' is a prime. Since $q''\equiv 1 \pmod 3$, we must have $q''=p$, which is impossible since $q''=(p+1)/2$. Thus n cannot be divisible by q, and this contradiction completes the proof.

In [3] Kanold gives several necessary conditions for the odd integer n given by

$$n = p^{\alpha} q_1^{2\beta} q_2^{2} \cdots q_t^{2}$$

to be perfect. We have shown that n is not perfect if $\beta\equiv 1 \pmod 3$. We shall now show that we may drop this requirement if we impose a condition on q_1. In particular, we shall show that n is not perfect if $q_1\equiv 2 \pmod 3$. We continue to assume, of course, that $n\not\equiv 0 \pmod 3$. Assume that n is perfect. Then $q_i \equiv 2 \pmod 3$ for $i=2,\cdots,t$. If $i\neq 2$, q_i cannot divide $\sigma(q_2^2)$. This is a result of the fact that $\sigma(q_2^2)=f_3(q_2)$, where $f_3(x)$ is the third cyclotomic polynomial, and if $f_3(x)\equiv 0 \pmod{q_i}$ has a solution, then $q_i\equiv 1 \pmod 3$ [4, p. 164]. Thus, $\sigma(q_2^2)=p^m$, and by a lemma due to A. Brauer [1], $m=1$. The same is true of $\sigma(q_i^2)$ for $i=3,\cdots,t$. Herein lies a contradiction. For, $t>3$, and so, even though $q_2\neq q_3$, we have $\sigma(q_2^2)=\sigma(q_3^2)$.

The author wishes to express his thanks to the referee for his comments and suggestions.

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THE CENTROID OF ANALYTIC MAPPINGS

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Let $f(z)=u+iv=\sum_{n=1}^{\infty} a_n z^n$ be regular for $|z|=|x+iy|\leqq R$. It is well known that the area of the domain D corresponding to $|z|\leqq r<R$ (multiply-covered parts being counted multiply) is given [2] by

$$A=\iint_{|z|\leqq r} dudv=\iint\left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy=\int_0^r\int_0^{2\pi}|f'(re^{i\theta})|^2 rdrd\theta$$

$$=\pi\sum_{n=1}^{\infty} n|a_n|^2 r^{2n}.$$

Let $(\bar{u},\bar{v})$ represent the centroid of the domain D. Following the usual definition (multiply-covered areas may be avoided if, for example, one restricts oneself to schlicht functions [1]), we have

$$A\bar{u}=M_v=\iint ududv; \qquad A\bar{v}=M_u=\iint vdudv; \qquad |z|\leqq r.$$

Writing $u=(f+\bar{f})/2$, $v=(f-\bar{f})/2i$, we obtain

$$M_v=\frac{1}{2}\int_0^r\int_0^{2\pi}[f(re^{i\theta})+\bar{f}(re^{i\theta})]\,|f'(re^{i\theta})|^2 rdrd\theta,$$

$$M_u=\frac{1}{2i}\int_0^r\int_0^{2\pi}[f(re^{i\theta})-\bar{f}(re^{i\theta})]\,|f'(re^{i\theta})|^2 rdrd\theta.$$

Substituting

$$f(re^{i\theta})=a_1 re^{i\theta}+a_2 r^2 e^{2i\theta}+\cdots,$$

$$f'(re^{i\theta})=2a_2 re^{i\theta}+3a_3 r^2 e^{2i\theta}+\cdots,$$

$$\bar{f}(re^{i\theta})=\bar{a}_1 re^{-i\theta}+\bar{a}_2 r^2 e^{-2i\theta}+\cdots,$$

and carrying out the integrations, we obtain

$$M_v=\frac{\pi}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} j(a_i a_j\bar{a}_{i+j}+\bar{a}_i\bar{a}_j a_{i+j})r^{2(i+j)},$$

$$M_u=\frac{\pi}{2i}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} j(a_i a_j\bar{a}_{i+j}-\bar{a}_i\bar{a}_j a_{i+j})r^{2(i+j)}.$$

These are equivalent to

$$M_v=\pi\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} j\,\mathrm{Re}\,(a_i a_j\bar{a}_{i+j})r^{2(i+j)},$$

$$M_u = \pi \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j \operatorname{Im}(a_i a_j \bar{a}_{i+j}) r^{2(i+j)}.$$

Thus, if we assume the convergence of the numerators, the formula for the centroid of the domain D becomes

$$\text{(1)} \qquad \begin{aligned} \tilde{u} &= \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j \operatorname{Re}(a_i a_j \bar{a}_{i+j}) r^{2(i+j)} \right\} \Big/ \left\{ \sum_{n=1}^{\infty} n |a_n|^2 r^{2n} \right\}, \\ \tilde{v} &= \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j \operatorname{Im}(a_i a_j \bar{a}_{i+j}) r^{2(i+j)} \right\} \Big/ \left\{ \sum_{n=1}^{\infty} n |a_n|^2 r^{2n} \right\}. \end{aligned}$$

If the mapping function $f(z)$ is written in the form $f(z) = a_0 + \sum_1^{\infty} a_n z^n$, apply a translation and obtain*

$$\tilde{u} = \operatorname{Re}(a_0) + \text{the fraction in formula (1) above,}$$

$$\tilde{v} = \operatorname{Im}(a_0) + \text{the fraction in formula (1) above.}$$

Let us now consider a special case of the locus $C(r)$ of the centroid $(\tilde{u}, \tilde{v})$ as a function of r. Without loss of generality take $a_0 = 0$ and let the mapping function be $f(z) = \sum_1^{\infty} a_n z^n$. The locus $C(r)$ will then pass through the origin. When is $C(r)$ a straight line through the origin? Obviously, if all the coefficients a_n are real, then from equation (1) we have $\operatorname{Im}(a_i a_j \bar{a}_{i+j}) = 0$ for all values of i and j, and thus $\tilde{v} = 0$ so that $C(r)$ coincides with a portion of the real axis. Thus a sufficient condition for $C(r)$ to be a straight line through the origin is that all the coefficients of $f(z)$ be real. This sufficiency condition, of course, is also obvious from the principle of reflection. A moment's thought will also show that the principle of reflection implies that the (essential) reality of the coefficients is also a necessary condition that $C(r)$ be a straight line through the origin. We now proceed to prove this result directly from equation (1). In fact we prove the following theorem:

THEOREM. *Let* $f(z) = \sum_0^{\infty} a_n z^n = \sum_0^{\infty} |a_n| e^{i\theta_n} z^n$ *be regular for* $|z| \leqq R$. *Let* $C(r)$ *be the locus of the centroid (given by* (1)*) of the domain* D *corresponding to* $|z| = r < R$. *Then* $C(r)$ *is a straight line if, and only if, all the coefficients* a_n *are (essentially) real.*

Proof. There results no loss of generality if we assume (a) $a_0 = 0$, by applying a translation, (b) $a_1 \geqq 0$ (*i.e.* $\theta_1 = 0$), by applying the rotation $f(ze^{-i\theta_1})$, and (c) $\alpha = 0$, by applying the rotation $e^{-i\alpha} f(z)$. The necessary and sufficient condition that $C(r)$ be a straight line through the origin with inclination $\alpha = 0$ is that $\tilde{v} = 0$ for all $0 \leqq r < R$. Thus from (1), with a slight change in notation, we obtain

* Although the formulas for the area of the domain D and for the length of arc of the curve bounding the domain D are found in most textbooks on the subject, equations (1) are not to be found in readily available textbooks.

$$\text{(2)} \qquad \operatorname{Im} \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} j(a_j a_{n-j})\bar{a}_n r^{2n} = 0; \qquad 0 \leqq r < R,$$

$$\operatorname{Im} [a_1^2 \bar{a}_2 r^4 + (3a_1 a_2)\bar{a}_3 r^6 + (4a_1 a_3 + 2a_2^2)\bar{a}_4 r^8 + \cdots] = 0.$$

Equation (2) is satisfied if, and only if,

$$\text{(3)} \qquad \phi = \arg \sum_{j=1}^{n-1} j a_j a_{n-j} \bar{a}_n \equiv \delta\pi; \qquad \delta = 0, 1; n = 2, 3, \cdots.$$

For $n=2$ equation (3) yields $\arg (a_1^2 \bar{a}_2) \equiv \delta\pi$, or $2\theta_1 - \theta_2 \equiv \delta\pi$, or $\theta_2 \equiv 2\theta_1 + \delta\pi \equiv \delta\pi$. Similarly, for $n=3$ we obtain $\arg (3a_1 a_2 \bar{a}_3) = \theta_1 + \theta_2 - \theta_3 \equiv \delta\pi$, or $\theta_3 \equiv \delta\pi$. Thus the relation $\theta_k \equiv \delta\pi$ holds for $k=2, 3$. By induction, assume $\theta_k \equiv \delta\pi$, $k = 1, \cdots, n-1$. Then, $\arg (j a_j a_{n-j}) = \theta_j + \theta_{n-j} \equiv \delta\pi + \delta\pi \equiv \delta\pi$, that is, each term of the sum $\sum j a_j a_{n-j}$ has the same argument, namely $\delta\pi$. Substituting in (4), we obtain $\phi = \delta\pi - \theta_n \equiv \delta\pi$, or $\theta_n \equiv \delta\pi$. Hence it follows that $\theta_n \equiv \delta\pi$, $n = 2, 3, \cdots$. Therefore, the coefficients of $f(z)$ are all real. This completes the proof of the theorem.

As a generalization of the above theorem, we state that if $f(z)$ has the form $f(z) = \sum_1^{\infty} |a_n| e^{i\theta_n} z^n$, and α is arbitrary, then $C(r)$ is a straight line through the origin with inclination α if, and only if, any one of the following four equivalent conditions is true:

(a) $e^{-i\alpha} f[z e^{i(\alpha - \theta_k)/k}] \equiv$ function with all real coefficients,

(b) $\theta_n = [n\theta_k - \alpha(n-k)]/k; \qquad n = k, k+1, \cdots,$

(c) $\theta_n = (n-k)\theta_{k+1} - (n-k-1)\theta_k; \qquad n = k, k+1, \cdots,$

(d) $\operatorname{Im} [a_n a_k^{(n-k-1)} \bar{a}_{k+1}^{(n-k)}] = 0; \qquad n = k, k+1, \cdots.$

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CHARACTERISTIC ROOTS OF SEMI-MAGIC SQUARE MATRICES

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In this note we consider the matrices formed by the magic and semi-magic squares. A square matrix $A = (a_{rs})$ of order n is called a magic square matrix if

$$\text{(1)} \qquad \sum_{r=1}^{n} a_{rs} = \sum_{r=1}^{n} a_{sr} = S(A), \text{ for } s = 1, 2, \cdots, n; \text{ and}$$

$$\text{(2)} \qquad \sum_{r=1}^{n} a_{rr} = \sum_{r=1}^{n} a_{r,n-r+1} = S(A).$$

However, if only (1) holds, A is said to be a semi-magic square matrix, and,

following Weiner, will be called an S-matrix of order n. In [4] Weiner has considered the matrix algebra R_n of S-matrices of order n and has also determined the structure of the algebra R_n. The main purpose of this note is to determine the bounds for the characteristic roots of S-matrices. Throughout this note the entries of the S-matrices lie in an ordered field of characteristic zero.

We first prove the following:

THEOREM 1. *If A is in G', the set of all nonsingular S-matrices of order n, then A^{-1} is also in G', and $S(A^{-1}) = (S(A))^{-1}$.*

Proof. To each element A of G', there exists a matrix $B \equiv (b_{ij}) = A^{-1}$, such that $AB = BA = I$, the identity matrix. Considering $AB = I$, we have $\sum_{r=1}^{n} a_{ir}b_{rj} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, which equals 1 when $i = j$ and equals 0 when $i \neq j$. *i.e.*, $\sum_{i=1}^{n}\sum_{r=1}^{n} a_{ir}b_{rj} = \sum_{i=1}^{n} \delta_{ij}$, or, $\sum_{r=1}^{n} b_{rj} \sum_{i=1}^{n} a_{ir} = 1$, or, $S(A)\sum_{r=1}^{n} b_{rj} = 1$. Therefore, $\sum_{r=1}^{n} b_{rj} = 1/S(A)$, a fixed quantity, for $j = 1, 2, \cdots, n$.

Similarly, proceeding with $BA = I$, it can be seen that $\sum_{j=1}^{n} b_{rj} = 1/S(A)$, for $r = 1, 2, \cdots, n$.

Thus $B = A^{-1}$ belongs to G' and is such that $S(A^{-1}) = \{S(A)\}^{-1}$. This completes the proof.

It may be observed here that the set G' of all nonsingular S-matrices of order n is a non-abelian multiplicative group. Further, it can be observed that every element A, distinct from the zero element (null matrix), belonging to G, the set of all S-matrices of the same order, generates a cyclic group of infinite order with addition as the rule of combination.

We shall now prove a theorem which gives an upper bound for the absolute value of any characteristic root of an S-matrix, A, with positive numbers as its elements.

THEOREM 2. *If λ is any characteristic root of A, then*

$$|\lambda| \leqq S(A), \tag{3}$$

i.e., the absolute value of λ is not greater than the sum of the elements of A along any row or column.

Proof. By a well known theorem of I. Schur, [3], there exists an orthogonal unitary matrix $U = (u_{ij})$ which transforms A into a triangular matrix T. The principal diagonal of T consists of the characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, not necessarily all distinct. Then from

$$T = UAU^*, \quad \text{and} \quad UU^* = U^*U = I,$$

it follows that $t_{ij} = \sum_{r,s=1}^{n} u_{ir}a_{rs}\bar{u}_{js}$, where t_{ij} is the element of T in the (i, j)th place.

That is, the elements of T are of the form $\sum_{r,s=1}^{n} a_{rs}x_r\bar{x}_s$, [1, p. 150], where

the set of (complex)numbers $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$ is such that $\sum_{i=1}^{n} x_i \bar{x}_i = 1$.

Let the absolute values of x_i be denoted by ξ_i $(i = 1, 2, \cdots, n)$. Then the set $(\xi_1, \xi_2, \cdots, \xi_n)$ is the set of real numbers such that $\sum \xi_i^2 = 1$. Also, since ξ_i, ξ_j are all real, $\xi_i \xi_j \leqq \frac{1}{2}(\xi_i^2 + \xi_j^2)$. Hence, $|t_{ii}| = |\sum_{r,s=1}^{n} a_{rs} x_r \bar{x}_s| \leqq \sum_{r,s} |a_{rs}| \cdot |x_r| \cdot |\bar{x}_s| = \sum_{r,s} a_{rs} \xi_r \xi_s \leqq \frac{1}{2} \sum_{r,s} a_{rs}(\xi_r^2 + \xi_s^2) = \frac{1}{2}[\sum_r \xi_r^2 \sum_s a_{rs} + \sum_s \xi_s^2 \sum_r a_{rs}] = \frac{1}{2}[S(A) \sum_r \xi_r^2 + S(A) \sum_s \xi_s^2] = S(A)$.

Therefore, $|t_{ii}| \leqq S(A)$. Hence, $|\lambda_i| \leqq S(A)$, $i = 1, 2, \cdots, n$. This establishes the theorem.

We now generalize the above theorem in the following form:

THEOREM 3. *Let $r(x) = f_1(x)/f_2(x)$ be a rational function of the scalar indeterminate x and A be an S-matrix, with positive numbers as its elements, such that $f_2(A)$ is nonsingular. If λ is a characteristic root of A, then for any characteristic root $r(\lambda)$ of $r(A)$,*

$$|r(\lambda)| \leqq r\{S(A)\}. \tag{4}$$

Proof. Since the S-matrices of the same order form an algebra (Theorem 1, [4]), $f_1(A)$ and $f_2(A)$ are S-matrices, the latter being nonsingular. It may further be verified that $S\{f_1(A)\} = f_1\{S(A)\}$, and $S\{f_2(A)\} = f_2\{S(A)\}$, whence $S\{r(A)\} = S\{f_1(A)(f_2(A))^{-1}\} = S\{f_1(A)\} \cdot S\{(f_2(A))^{-1}\} = S\{f_1(A)\} \cdot \{S(f_2(A))\}^{-1} = f_1\{S(A)\}/f_2\{S(A)\} = r\{S(A)\}$.

Thus $r(A) = f_1(A)/f_2(A)$ is an S-matrix. Also, by a well known theorem of Frobenius [2, pp. 22, 23], if λ is a characteristic root of A, $r(\lambda)$ is a characteristic root of $r(A)$. In order to see that $r(\lambda)$ is defined, we observe that since the matrix $f_2(A)$ is nonsingular, no characteristic root of $f_2(A)$ is zero. That is, $f_2(\lambda) \neq 0$, for it is a characteristic root of $f_2(A)$.

Now, applying (3), we have $|r(\lambda)| \leqq r\{S(A)\}$, and the theorem is proved.

Finally, I am indebted to Professor S. M. Shah for his help in the preparation of this paper and to the referee for helpful criticism.

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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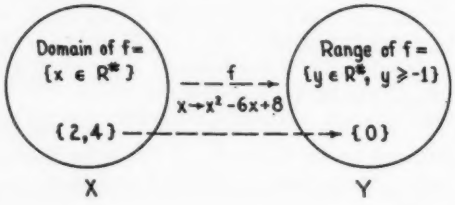
AN EARLY INTRODUCTION TO THE MAPPING CONCEPT

V. O. McBRIEN, College of the Holy Cross

It is evident from the development in Part I of *Universal Mathematics** that the concept of mapping should be introduced at an early stage in mathematical training. Variations of the following method have been used by a number of teachers early in the first semester of their freshman courses.

The graph of a familiar polynomial such as $y=x^2-6x+8$ is made by the usual method which the student has learned in high school. A function is then defined as a triple $(X, Y; f)$ consisting of two sets X and Y and a relation f in the Cartesian product $X \times Y$ such that no $x \in X$ is the first element of two pairs (x, y) and (x, y'). The quadratic polynomial x^2-6x+8 is now considered in the light of this definition. In this case we have two sets of real numbers X and Y where X is mapped onto Y as follows:



It is pointed out that by the law $f: x \to x^2-6x+8$ we have performed a mapping process by which the open set of all real numbers is mapped onto the half-open set of real numbers equal to or greater than -1. With a slight apology to a future mathematics major, the subset $\{2, 4\}$ of X, which is mapped into $\{0\}$ by f, may be called the kernel of the map. The student is happy to tie this notion in with his idea of the roots of an equation.

Other simple examples of algebraic functions may be done in the same way, and this method is helpful in the introduction to the transcendental functions. For example, by using the map $x \to \log x$, the student sees that we have a homomorphic mapping of the multiplicative set of positive real numbers onto the additive set of real numbers.

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* Universal Mathematics, Part I, Functions and Limits, Student Union Book Store, Univ. of Kansas, 1954.

SYNTHETIC APPROACH TO THE THEORY OF THE ENVELOPE

A. R. AMIR-MOÉZ, University of Idaho

DEFINITION. *Let* (C) *be a curve varying according to a certain rule. If there is a curve* (γ) *tangent to* (C) *in all its positions, then* (γ) *is called the envelope of* (C).

The basic approach to the problem of finding envelopes is similar to that used in differential geometry. In this paper, instead of the analytic treatment, we give the synthetic solutions of a few problems. In some cases, where the equation of the envelope is quite complicated, it is easier to describe the envelope synthetically.

Consider (C_1) and (C_2), two positions of (C), so that $(C_1)\cap(C_2)=K_1$. Let (C_2) approach (C_1). Then K_1 approaches K which is called the characteristic point. The locus of K is the envelope of (C) if it exists.

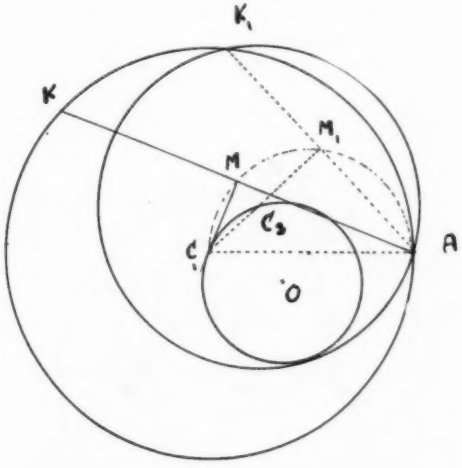

FIG. 1

Problem I. Let (O) be a fixed circle with center O, and A a fixed point in the plane of (O). Circle (C), in the plane of (O), passes through A and its center C moves on (O). Find the envelope of (C).

Solution. Let (C_1) and (C_2) be two positions of (C) (Fig. 1). Clearly C_1C_2 is the perpendicular bisector of AK_1, the common chord of (C_1) and (C_2). It is observed that M_1, the midpoint of AK_1, is on the circle with diameter AC_1. As C_2 tends to C_1, C_1C_2 tends to the tangent C_1M to (O) at C_1, and M_1 approaches M, the point of intersection of C_1M and the circle with diameter AC_1. But AM intersects (C_1) at K which is the limit of K_1. Therefore the locus of K is the envelope.

This locus is easily described as follows:

$AK=2AM$, and M is the vertex of a right angle with one side passing through A and the other side tangent to (O).

Draw a perpendicular through O to AM, (Fig. 2). The foot of the perpendicular, N, is on the circle with diameter OA, and NM is equal to the radius of (O). This describes a limaçon L. Therefore the locus of K is a limaçon, homothetic of L with ratio 2.

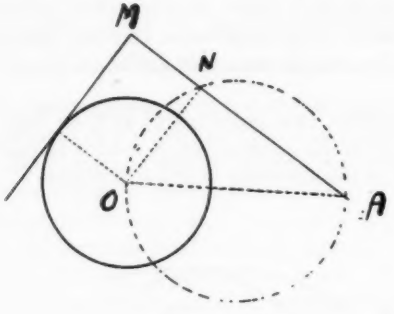

FIG. 2

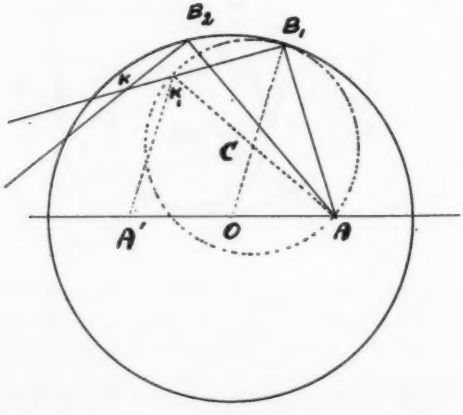

FIG. 3

Problem II. The circle (O) with center O, and the point A, on the plane of (O), inside (O), are fixed. The envelope of the second side of a right angle is desired such that its first side passes through A and its vertex moves on (O), (Fig. 3).

*Solution.** Let AB_1K and AB_2K be two positions of the right angle where K

* This solution is due to Dr. M. Hachtroudi, Professeur à l'Université Teheran.

is the point of intersection of the second sides of these angles. It is obvious that A, B_1, B_2, K are on a circle. As B_2 approaches B_1 the circle through A, B_1, B_2, K tends to the circle (C) through A, B_1 and tangent to (O) at B_1. This circle intersects B_1K at K_1, *i.e.* K_1 is the characteristic point.

It is clear that $K_1A = 2CB_1$. If we draw a line through K_1 parallel to CO, this line intersects OA at A_1, and $K_1A_1 = 2OC$. Therefore $K_1A + K_1A_1 = 2OB_1$, *i.e.* the envelope is an ellipse.

Problem III. Find the envelope of circle (C), with the center C on a given hyperbola with foci F_1 and F_2, and tangent to the fixed circle (F_1) with center F_1.

Solution. Let (C_1) and (C_2), two positions of (C), intersect at K_1 and H_1, (Fig. 4). Then K_1H_1 is perpendicular to C_1C_2. Let C_2 tend to C_1. Then C_1C_2 becomes tangent C_1T to the hyperbola at C_1, and H_1 tends to H, the point of

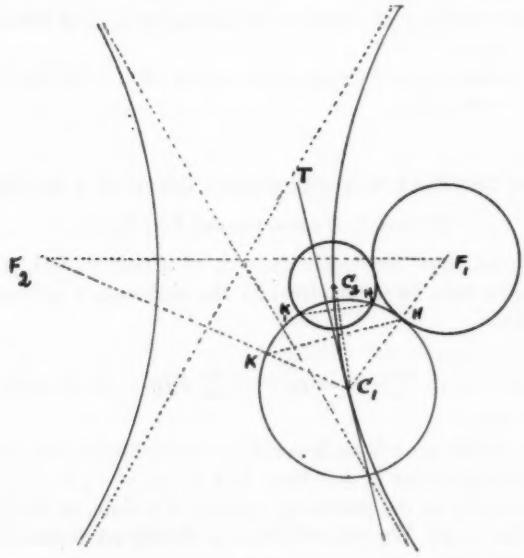

FIG. 4

tangency of (C_1) and (F_1). Therefore K_1 approaches K, the point of intersection of the line through H perpendicular to C_1T. So K is the characteristic point.

Now it is observed that $F_2K = C_1F_2 - C_1K = C_1F_2 - (C_1F_1 - HF_1) = (C_1F_2 - C_1F_1) - HF_1 = \text{constant}$. Therefore the envelope is the circle with center F_2 and radius F_2K.

Examples:

(1) Find the envelope of the second side of the right angle such that its first side passes through a fixed point and its vertex moves on a fixed straight line.

(2) Solve (1) if the angle is equal to a fixed angle α, instead of the right angle.
(3) Find the envelope of the free side of a right angle whose vertex is on a fixed circle and the other side passes through a fixed point outside of the fixed circle.
(4) The same as (3) except use an angle equal to α.
(5) Solve Problem II for angle α instead of right angle.
(6) Solve Problem III where an ellipse or a parabola is used instead of a hyperbola.
(7) Find the envelope of circles with the center on a fixed circle and tangent to another fixed circle.
(8) Find the envelope of circles with center on a fixed circle and tangent to a fixed line.
(9) Find the envelope of circles with center on a fixed line and tangent to a fixed circle.
(10) Find the envelope of circles with center on a fixed line and tangent to a given curve (C).

A PASCAL TRIANGLE FOR THE COEFFICIENTS OF A POLYNOMIAL

RICHARD A. MILLER, Convair, Fort Worth

It is well known that the coefficients, c_i, of a polynomial, $f(x)=x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n$, may be determined by the elementary symmetric functions of its zeros, r_i, *i.e.*,

$$\sum_i^n r_i = -c_1,\ \sum r_1r_2 = c_2,\ \cdots,\ \sum r_1r_2\cdots r_n = (-1)^n c_n.$$

The purpose of this note is to develop an algorithm for the formation of the coefficients from a given set of numbers $\{r_i\}$, $i=1, \cdots, n$.

We denote the sum of the products, taken j at a time, of the first k elements of the set by $S(r_i, j, k)$. We observe that a simple recursion formula can be established by induction,

$$(1)\qquad S(r_i, j, k) = S(r_i, j, k-1) + r_kS(r_i, j-1, k-1) \qquad \text{for } j>1,\ k>1.$$

We define $S(r_i, 0, k)=1$ for $k>0$, and $S(r_i, j, k)=0$, for $k<j$.

Verification for $k=2$ is immediate, for we have $S(r_i, 1, 2)=S(r_i, 1, 1)+r_2S(r_i, 0, 1)=r_1+r_2$, and $S(r_i, 2, 2)=S(r_i, 2, 1)+r_2S(r_i, 1, 1)=0+r_2r_1$.

We assume validity for k and deduce it for $k+1$. We note that

$$S(r_i, 1, k+1) = S(r_i, 1, k) + r_{k+1}S(r_i, 0, k) = (r_1+\cdots+r_k)+r_{k+1},$$

$$r_{k+1}S(r_i, k, k) = S(r_i, k+1, k+1).$$

For $1<j<k+1$, we consider the expression

$$S(r_i, j, k) + r_{k+1}S(r_i, j-1, k). \tag{2}$$

This is the sum of products of the elements $r_1, r_2, \cdots, r_{k+1}$ taken j at a time. $S(r_i, j, k)$ clearly contains all the terms not containing r_{k+1}. Those terms containing r_{k+1} are the products of r_{k+1} by the product of $j-1$ elements taken from $r_1, \cdots, r_k$, hence the sum of all the terms is $r_{k+1}S(r_i, j-1, k)$. Thus (2) is the sum of all products which can be formed by taking j factors at a time from $r_1, r_2, \cdots, r_{k+1}$, *i.e.*, $S(r_i, j, k+1)$.

Thus the recursion formula suggests an arrangement such that $S(r_i, j, k)$ will be the element in the jth row and the kth column of a matrix. We use the scheme shown in Table 1. If the elements of the jth row are multiplied by $(-1)^j$, the kth column contains the coefficients $S(r_i, 0, k)$, $-S(r_i, 1, k)$, $S(r_i, 2, k)$, $\cdots$, $(-1)^kS(r_i, k, k)$ of a polynomial with zeros, $r_1, \cdots, r_k$.

TABLE 1

	k	1	2	k	n
j	r	r_1	r_2	r_k	r_n
0					
1					
j				$S(r_i, j, k)$	
n					

TABLE 2

	k	1	2	3	4
j	r_k	2	3	5	7
0		1	1	1	1
1		2	5	10	17
2			6	31	101
3				30	247
4					210

For the set $\{r_i\} = \{2, 3, 5, 7\}$, we have the array, shown in Table 2. We are able to secure the following polynomials $P_k(x)$ with the zeros as indicated:

$P_k(x)$	Zeros
$P_1(x) = x - 2$	2
$P_2(x) = x^2 - 5x + 6$	2, 3
$P_3(x) = x^3 - 10x^2 + 31x - 30$	2, 3, 5
$P_4(x) = x^4 - 17x^3 + 101x^2 - 247x + 210$	2, 3, 5, 7.

This process can be applied for the factorial polynomial,

$$P^n(x) = x(x-1)(x-2)\cdots[x-(n-1)] = S_0^n x^n + S_1^n x^{n-1} + \cdots + S_{n-1}^n x,$$

where S_j^n are the Stirling numbers of the first kind, which satisfy the recursion relation $S_j^{n+1} = S_j^n - nS_{j-1}^n$. For example, an equation with roots $1, \cdots, 10$ will be $x^{10} - 55x^9 + 1320x^8 - 18150x^7 + 157773x^6 - 902055x^5 + 3416930x^4 - 8409500x^3 + 12753576x^2 - 10628640x + 3628800 = 0$.

We remark that if all the roots of the polynomial are -1, *i.e.*, $f(x) = (x+1)^n$, the kth column will give the binomial coefficients, for which ${}_kC_j = {}_{k-1}C_{j-1} + {}_{k-1}C_j$.

INTERPRETATIONS OF THE PEANO POSTULATES

M. D. Darkow, Hunter College

When students first deal with an abstract postulate system, their understanding profits from exercises involving models that satisfy the system. This paper contains suggestions for such exercises in connection with the Peano postulates for the natural numbers.

The Peano postulates are given in terms of an undefined concept, called a natural:

1) To every natural n, there corresponds a natural n', called the consequent of n.

2) There exists a natural, denoted by 1, which is not a consequent.

3) Naturals having equal consequents are equal.

4) If a set of naturals contains 1, and contains the consequent of every natural in the set, it is the set of all naturals.

The interpretations or models of these postulates utilize the students' prior knowledge.

Model A: The numbers $a, a+d, a+2d, \cdots$ of an arithmetic progression (in which a, d are real numbers, $d \neq 0$) satisfy the Peano postulates if a is the Peano 1, and n' is defined as $n+d$.

Model B: The numbers $a, ar, ar^2, \cdots$ of a geometric progression (in which a, r are arbitrary real numbers, $0 \neq a$, $0 < r \neq 1$) satisfy the Peano postulates if a is the Peano 1, and $n' = nr$. (A simple adjustment takes care of the case $0 > r \neq -1$.)

Theorem. *There is one and only one operation* $\circ$ *combining two naturals* b *and* c *into a natural, such that* $b \circ 1 = b'$ *and* $b \circ c' = (b \circ c)'$. *This operation is called Peano addition and is denoted by* $(+)$. *For it*

$$b(+)1 = b' \text{ and } b(+)c' = [b(+)c]'. \tag{1}$$

The students are then asked to find the proper definitions for Peano addition in models A and B. Since it is no secret to them that the natural numbers are defined by the Peano postulates, they might proceed as follows:

A. If $x = a+(h-1)d$, $y = a+(k-1)d$, then $z = x(+)y$ should equal $a+(h+k-1)d$. Elimination of h and k produces

$$x\,(+)\,y = x + y - a + d. \tag{2}$$

The students then prove that this definition actually has properties (1).

B. If $x = ar^{h-1}$, $y = ar^{k-1}$, then $z = ar^{h+k-1}$ should define $x(+)y$. Hence $xyr/a = ar^{h+k-1} = z$ and

$$x\,(+)\,y = rxy/a. \tag{3}$$

This definition must be proved to satisfy (1).

Theorem. *There is one and only one operation* $*$ *combining two naturals* b *and* c *into a natural, such that* $b * 1 = b$ *and* $b * c' = b * c(+)b$. *This operation is*

called Peano multiplication and is denoted by $(\cdot)$. *Hence*

$$b\,(\cdot)\,1 = b \quad \text{and} \quad b\,(\cdot)\,c' = b\,(\cdot)\,c\,(+)\,b. \tag{4}$$

The students then find the proper definitions of Peano multiplication for models A and B.

A: $x(\cdot)y=[a+(h-1)d](\cdot)[a+(k-1)d]$ should equal $z=[a+(hk-1)d]$. Elimination of h and k yields

$$z = x\,(\cdot)\,y = a - d + (x - a + d)(y - a + d)/d. \tag{5}$$

This definition must be shown to satisfy (4).

B: $x(\cdot)y=ar^{h-1}(\cdot)ar^{k-1}$ should equal $z=ar^{hk-1}$. Elimination of h and k yields $\log_r rz/a=\log_r rx/a\cdot\log_r ry/a$. Hence

$$z = x\,(\cdot)\,y = ar^E, \text{ where } E = (1 + \log_r x/a)(1 + \log_r y/a) - 1. \tag{6}$$

The students then show that this definition has property (4).

Suitable definitions of the Peano order relation may also be asked in A and B.

When the categorical nature of the postulates has been established, the students are asked to demonstrate the isomorphism of models A and B.

Any sequence of distinct complex numbers may be similarly treated.

METHOD OF UNDETERMINED COEFFICIENTS

M. S. Webster, Purdue University

Consider the special linear differential equation of order n, $(D-a)^n y=x^k e^{ax}$ where n is a positive integer, k is a nonnegative integer, a is any real constant, and D is the operator d/dx. The usual rule* given for finding a particular solution by the method of undetermined coefficients is to multiply $e^{ax}(A_0x^k+A_1x^{k-1}+\cdots+A_{k-1}x+A_k)$ by an appropriate power of x (x^n in this case) and then to determine the constants $A_0, A_1, \cdots, A_k$ so that $x^n e^{ax}(A_0x^k+\cdots+A_k)$ is a solution. This procedure will do, but, because of the special nature of the left member of this differential equation, it may be greatly simplified in this case.

Actually the equation has a solution of the form $Ax^{n+k}e^{ax}$ where A is a constant. This may be shown by means of the shifting rule $e^{-ax}P(D)y=P(D+a)\cdot(e^{-ax}y)$ where $P(D)$ is a polynomial in D. The differental equation becomes $D^n(e^{-ax}y)=x^k$, from which the result follows.

* Apparently first given by A. Cohen in *An Elementary Treatise on Differential Equations*, 1906.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1261. *Proposed by C. S. Ogilvy, Hamilton College*

If the sectorial area bounded by a curve and the radii vectores to any two points on the curve equals the area bounded by the curve, the x-axis, and the ordinates of the two points, find the curve.

E 1262. *Proposed by A. J. Goldman, National Bureau of Standards*

For which real values of A do all roots of $z^3-z^2+A=0$ obey $|z| \leqq 1$?

E 1263. *Proposed by W. B. Carver, Cornell University*

Let T_0 be any triangle none of whose angles is a multiple of 45°. The tangents to the circumcircle of T_0 at its vertices form a new triangle T_1; and repetition of this process gives an infinite sequence of triangles $\{T_n\}$. If the angles of T_0 in degrees are integers, show that for $n \geqq 2$, T_{n+12} is similar to T_n. Is there a similar theorem for the case when the angles are rational in degrees?

E 1264. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If an interior point P of a tetrahedron $ABCD$ is projected orthogonally into A', B', C', D' on the planes of the faces BCD, CDA, DAB, ABC, and if the areas of these faces are denoted by A, B, C, D, show that

$$A(PA) + B(PB) + C(PC) + P(PD) \geqq 3[A(PA') + B(PB') + C(PC') + D(PD')].$$

E 1265. *Proposed by F. L. Wolf, Carleton College*

Is there a nonnegative, nontrivial, continuous function $f(x)$ such that

$$\int_0^x f(t)dt \geqq f(x)$$

for all x on $0 \leqq x \leqq 1$?

SOLUTIONS

A Surface of Revolution

E 1231 [1956, 577]. *Proposed by A. W. Walker, University of Toronto*

Show that $x^3+y^3+z^3-3xyz=a^3$ is a surface of revolution.

I. *Solution by E. W. Marchand, Eastman Kodak Company.* The equation may be written $(x+y+z)[3(x^2+y^2+z^2)-(x+y+z)^2]=2a^3$. Hence the intersection of this surface with any plane of the form $x+y+z=c$ lies on a sphere $x^2+y^2+z^2=$constant and so must be a circle having the normal to this plane from the origin as axis.

II. *Solution by Leonard Carlitz, Duke University.* One easily verifies the identity

$$x^3+y^3+z^3-3xyz=(x+y+z)[(x-y/2-z/2)^2+3(y-z)^2/4].$$

The transformation

$$x'=(x+y+z)/\sqrt{3},\quad y'=\sqrt{2/3}(x-y/2-z/2),\quad z'=(y-z)/\sqrt{2},$$

is orthogonal with determinant $+1$. Thus the equation of the surface in the new coordinate system becomes $3\sqrt{3}x'(y'^2+z'^2)=2a^3$. The surface is obtained by revolving the curve $3\sqrt{3}x'y'^2=2a^3$ about OX'.

Also solved by Norman Anning, A. P. Boblétt, T. Y. Chow, Hazel E. Evans, Edward Fleisher, L. R. Ford, Michael Goldberg, A. S. Hendler, Frank Herlihy, A. R. Hyde, I. M. Isaacs, J. B. Johnston, M. S. Klamkin, D. C. B. Marsh, John Rausen, L. A. Ringenberg, Robin Robinson, Azriel Rosenfeld, David Zeitlin, and the proposer. Late solution by Sidney Glusman.

Editorial Note. The surface has the shape of an infinite horn; it is asymptotic to the plane $x+y+z=0$ and to the axis of revolution $x=y=z$.

A Line Through the Orthocenter of a Triangle

E 1232 [1956, 577]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Arbitrary, parallel lines drawn through the vertices A, B, C of a triangle intersect the circumcircle in A', B', C'. Show that A'', B'', C'', the symmetrics of these points with respect to the midpoints of BC, CA, AB, respectively, lie on a line perpendicular to the parallel lines and passing through a fixed point of the triangle.

I. *Solution by John Rausen, New York University.* Let O be the circumcenter and H the orthocenter of triangle ABC, and let M be the midpoint of BC. Let AH intersect the circumcircle at H'. By a well known theorem, H' is the symmetric of H with respect to BC. Let H'' be the other end of the diameter through A. Then $H'H''$ is perpendicular to AH', hence parallel to BC, and chords BC, $H'H''$ have the same perpendicular bisector, OM. Thus H'' is the symmetric of H' with respect to OM. By composition of two symmetries about perpendicular lines, H'' is the symmetric of H with respect to M. Therefore lines $A'H''$ and $A''H$ are symmetric with respect to M, and hence are parallel. But $A'H''$ is perpendicular to AA' (since AH'' is a diameter). Therefore so is $A''H$.

Similarly $B''H$ is perpendicular to BB' and $C''H$ is perpendicular to CC'. This proves all at once that (1) A'', B'', C'' lie on a line m, (2) m is perpendicular to AA', BB', CC', and (3) m passes through H, a fixed point in triangle ABC.

II. *Solution by A. R. Hyde, West Hartford, Conn.* If the origin is taken at the center of the circumscribed circle, and the x-axis is taken parallel to the parallel lines through the vertices, then the vertices are

$$A:(r\cos\alpha, r\sin\alpha), \quad B:(r\cos\beta, r\sin\beta), \quad C:(r\cos\gamma, r\sin\gamma),$$

where α, β, γ are the angles between the positive x-axis and the radii to A, B, C respectively. A', B', C' are the mirrors of A, B, C across the y-axis. Hence A'', B'', C'' all have x-coordinate equal to $r(\cos\alpha+\cos\beta+\cos\gamma)$, and $A''B''C''$ is a straight line perpendicular to the parallel (horizontal) lines through the three vertices. The altitudes of triangle ABC intersect on this line, as can be shown analytically.

Also solved by J. W. Clawson, J. C. W. De la Bere, J. T. Humphrey, Josef Langr, D. C. B. Marsh, Beckham Martin, Robert Sibson, Sister M. Stephanie, and the proposer.

Clawson, De la Bere, and Sister Stephanie gave proofs in the complex plane. Clawson stated the following generalization: If $AB\cdots N$ is an n-gon inscribed in a circle (O), and if a set of parallel lines are drawn through the vertices $A, B, \cdots, N$ to intersect (O) at $A', B', \cdots, N'$, and if $A_1, B_1, \cdots, N_1$ are the mean centers of the polygons obtained by omitting in turn $A, B, \cdots, N$, then if $A'A_1, B'B_1, \cdots, N'N_1$ are extended $1/(n-2)$ of these lengths to $A'', B'', \cdots, N''$, these latter points lie on a straight line perpendicular to the parallel lines and passing through the point which lies on the line joining O to M, the mean center of the polygon, extended $2/(n-2)$ of the length of OM.

On p. 6 of his book *Parmi les belles figures de la géométrie dans l'espace*, the proposer has extended the problem to a tetrahedron and its circumsphere; here A'', B'', C'', D'', are the mirrors of A', B', C', D' in the centroids of the faces BCD, CDA, DAB, ABC, and A'', B'', C'', D'' are coplanar on a plane perpendicular to the given parallel lines and passing through the Monge point of the tetrahedron.

A Sequence of Triangles

E 1233 [1956, 578]. *Proposed by Joseph Andrushkiw, Seton Hall University*

Denote the sides and inradius of a triangle by a_0, b_0, c_0, and r_0. The points of contact form a new triangle whose sides and inradius are a_1, b_1, c_1, and r_1. Repeating the process one obtains the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{r_n\}$. Show that as $n\to\infty$

$$\lim r_n/a_n = \lim r_n/b_n = \lim r_n/c_n = \sqrt{3}/6.$$

I. *Solution by N. J. Fine, University of Pennsylvania.* Let A_n, B_n, C_n be the angles opposite a_n, b_n, c_n respectively, for $n\geq 0$. It is easy to see that $a_n/r_n = \cot(B_n/2)+\cot(C_n/2)$, and similarly for the other two ratios. Now $A_{n+1} = (\pi-A_n)/2$. An easy induction shows that $A_n=\pi/3+(-1/2)^n(A_0-\pi/3)\to\pi/3$, and similarly $B_n\to\pi/3$, $C_n\to\pi/3$. Hence a_n/r_n, b_n/r_n, c_n/r_n all converge to $2\cot(\pi/3)=2\sqrt{3}$, from which the required result follows.

II. *Solution by Leon Bankoff, Los Angeles, Calif.* Let A_i denote the vertices opposite a_i $(i=0, 1, \cdots, n)$. It is easily shown that the acute angle between two chords of contact of an incircle is equal to the arithmetic mean of the angles at the two adjacent vertices of the parent triangle. It follows that

$2A_n = B_{n-1} + C_{n-1}$, $2B_n = A_{n-1} + C_{n-1}$, and $|A_{n-1} - B_{n-1}| = 2|B_n - A_n|$.

Then, as $n \to \infty$, $|B_n - A_n| \to 0$. Similarly $|A_n - C_n| \to 0$ and $|C_n - A_n| \to 0$, with the result that A_n, B_n, C_n each tend toward 60° as $n \to \infty$. This leads at once to the desired conclusion.

Also solved by W. B. Carver, J. C. W. De la Bere, A. R. Hyde, I. M. Isaacs, J. B. Johnston, P. G. Kirmser, M. S. Klamkin, Sadia Makky, D. C. B. Marsh, R. M. Meisel, C. S. Ogilvy, D. S. Passman, John Rausen, Azriel Rosenfeld, R. E. Shafer, Michael Skalskyj, A. V. Sylwester, Chih-yi Wang, and the proposer. Late solution by J. W. Harter.

The convergence of the angles A_n, B_n, C_n to $\pi/3$ is quite rapid. Carver remarked that if we start with *any triangle whatever*, all the angles of triangle $A_7B_7C_7$ will differ from 60° by less than 1°.

Two Unique Sets of Numbers

E 1234 [1956, 578]. *Proposed by Leo Moser and J. R. Pounder, University of Alberta*

(a) Find 3 distinct positive integers, relatively prime in pairs, such that the sum of any two is divisible by the third. Prove that these integers are unique.

(b) Find 3 distinct positive integers such that the product of any two leaves a remainder of 1 on division by the third. Prove that the numbers are unique.

Solution by E. P. Starke, Rutgers University. First we consider three positive integers $x<y<z$ and note that if z is a divisor of $x+y$, then $z=x+y$ because $x+y<2z$. If also y divides $x+z(=2x+y)$, then y divides $2x$ which, with $2y>2x$, implies $y=2x$, whence $z=3x$.

(a) By the above, if the integers x, $2x$, $3x$ are relatively prime by pairs, they must be 1, 2, 3. (Note that we have not used the hypothesis that $y+z$ is a multiple of x.)

(b) Let the integers be $a<b<c$, and let $ab-1=cx$. Then x is an integer less than a. Now $x(ac-1)=a^2b-a-x$ divisible by b implies $a+x$ divisible by b; and similarly $b+x$ divisible by a. By the initial result above, we have $a=2x$, $b=3x$. Finally $ab-1=cx$ gives $6x^2-1$ divisible by x, whence $x=1$ and the required unique integers must be $a=2$, $b=3$, $c=5$.

Also solved by W. J. Blundon, Anne W. and Jean M. Calloway (jointly), Leonard Carlitz, Monte Dernham, Underwood Dudley, Hazel E. Evans, N. J. Fine, Virginia S. Hanly and Ernest Kanning (jointly), Roger Hou and Lincoln Teng (jointly), J. B. Johnston, Joe Lipman, D. C. B. Marsh, P. J. McCarthy, Azriel Rosenfeld, E. D. Schell, R. E. Shafer, and the proposers. Late solutions by D. A. Breault and J. W. Harter.

A Uniformly Convergent Series

E 1235 [1956, 578]. *Proposed by D. S. Greenstein, University of Pennsylvania*

Let $f_0(x)$ be bounded and integrable over $a \leqq x \leqq b$, and let

$$f_n(x) = \int_a^x f_{n-1}(t)\,dt, \qquad n = 1, 2, \cdots, \qquad a \leqq x \leqq b.$$

Evaluate $\sum_{n=1}^{\infty} f_n(x)$.

I. *Solution by T. S. Chihara, Seattle University.* $f_n(x)$ being an n-fold iterated integral of $f_0(x)$, we have

$$f_n(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f_0(t)dt, \qquad |f_n(x)| \leqq \frac{(x-a)^{n-1}}{(n-1)!} \max |f_0(x)|,$$

for $a \leqq x \leqq b$. The interchange of summation and integration thus being justified, we have

$$\sum_{n=1}^{\infty} f_n(x) = \int_a^x e^{x-t} f_0(t)d = e^x \int_a^x e^{-t} f_0(t)dt.$$

II. *Solution by P. G. Kirmser, Kansas State College.* It may be shown (as in Solution I) that the series converges uniformly, to $y(x)$, say. We then find that $y'(x) = y(x) + f_0(x)$. Solving this differential equation gives

$$y(x) = e^x \int_a^x e^{-t} f_0(t)dt.$$

Also solved by Thomas Erber, A. S. Hendler, A. R. Hyde, J. B. Johnston, M. S. Klamkin, A. E. Livingston, Marshall Luban, C. M. Luján, Y. L. Luke, D. C. B. Marsh, R. M. Meisel, C. S. Ogilvy, F. D. Parker, John Rausen, Azriel Rosenfeld, R. E. Shafer, Michael Skalskyj, H. S. Valk, Albert Wilansky, L. K. Williams, David Zeitlin, and the proposer. Late solution by Sidney Glusman.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

4733. *Proposed by Albert Wilansky, Lehigh University*

Give an example of a compact Hausdorff space which is separable (has a dense sequence) but not completely separable (does not satisfy the second axiom of countability).

4734. *Proposed by George Brauer, University of Minnesota*

Construct an ordinary Dirichlet series $F(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \text{and} \quad \lim_{\sigma \to 0^+} |F'(\sigma + i\tau)| = \infty$$

for almost all points on the τ-axis ($s=\sigma+i\tau$). (A Taylor series with analogous properties is given in Rudin's paper, *On a problem of Bloch and Nevanlinna*, Proc. Amer. Math. Soc. vol. 6, 1955, pp. 202–204.)

4735. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i, and the common point of the lines A_iB_{i+1}, B_iA_{i+1} by C_{i+3}. Then the five lines A_iC_i have a point D in common.

4736. *Proposed by M. v. Ments, Jerusalem, Israel*

An encyclopaedia consists of N volumes, and in a certain town p persons originally possessed complete sets. But by some event, a lot of volumes of the various sets were destroyed at random, so that only n_i volumes remain in the possession of the ith person. If all the townspeople now pool their books, what is the probability that M volumes (out of the total N) are available somewhere in town?

4737. *Proposed by E. P. Starke, Rutgers University*

For any modulus m, let $V(m)$ be the number of values of $r(0\leq r\leq m-1)$ for which the congruence $x^2\equiv r \pmod m$ has at least one solution. Determine the form of $V(m)$.

SOLUTIONS

Disjoint Permutations

4647 [1955, 447]. *Proposed by James Munkres, Los Alamos Scientific Laboratory*

A permutation of the integers $1, \cdots, n$ is called an n-chain; two n-chains are disjoint if any two integers which are adjacent in one chain are not adjacent in the other. (The first integer is considered adjacent to the last for this purpose.) Does there exist, for each n, a collection containing $[(n-1)/2]$ mutually disjoint n-chains?

Editorial Notes. I. No solution has been received, but K. Eisemann points out that the problem is easy to solve for $n=p$, a prime, and that this solution is the basis for an article by H. P. Lawther, Jr., *An application of number theory to the splicing of telephone cables*, this MONTHLY, vol. 42, 1935, pp. 81–91 (also noted in Ore, *Number Theory and Its History*, pp. 302–310.)

In fact the least positive residues (mod p) of the numbers $1+rk$, ($r=0, \cdots, p-1$; $k=1, \cdots, (p-1)/2$) give a complete set. Furthermore, the set just described can be made into a satisfactory set for $n=p+1$ by inserting the number $(p+1)$ into each chain according to the rule: in the kth chain, for k odd, put $(p+1)$ between $(k+1)/2$ and $(2p-k+1)/2$; for k even, put $(p+1)$ between $(p+k+1)/2$ and $(p-k+1)/2$.

II. Experimentation with small values of n makes plausible an affirmative

answer to the proposer's question. Starting with a set for $n=n_1$, we form a set for $n=n_1+1$ by inserting the new number(n_1+1) into each chain of the old set. For n_1 odd, every number is adjacent once and only once to every other number and, provided n_1 is not large, it is fairly easy to choose from each chain a pair to be separated by the new number (n_1+1) in such a way that all the numbers are different. In fact, this may be done in many ways as n_1 increases. For n_1 even, each number touches all other numbers but one—which already provides pairs which can be used to form the additional chain needed for n_1+1; the remaining pairs come when (n_1+1) is inserted in a chain, thus separating two numbers which can be brought together again in the new chain to be formed.

No simple procedure has developed by which this can be systematized so as to provide a proof of our conjecture by induction. The question is still open and a proof (or counter-example) is desired.

The following set of n-chains for $n=17$ was developed step by step starting with $n=3$. (Letters are used instead of numerals for ease in transcribing.) The intermediate steps may be recaptured by dropping any number of the last letters and retaining only the required number of chains at the beginning. It will be noted that at no step do these sets follow the scheme outlined in I.

abcdefghijklmnopq	*bnloigjmqehcadfkp*	*cmgnkqdlhjboeafip*
dpmiekocjlqfnhagb	*hkiajpndogelcfmbq*	*hmkalpgqcnejofbid*
homanqilbeckgdjfp	*jqoapemdkbhflgcin*	

A Conjecture Concerning Primes

4683 [1956, 258]. *Proposed by Edgar Karst, Endicott, N. Y.*

Where fails the "interesting conjecture" (so called by D. H. Lehmer in a letter to the proposer, March 1952): If 2^p-1 is prime, then also $2^p-1+100$ is prime?

Solution by J. L. Selfridge, University of California, Los Angeles. For $p<20$, the conjecture is true, as can be checked by examining a table of primes. For each p in the accompanying table, we list the least factor f of 2^p+99. This table includes all values of $p>19$ for which 2^p-1 is known to be prime (R. M. Robinson, *Mersenne and Fermat numbers*, Proc. Amer. Math. Soc. vol. 5, 1954, p. 842.)

p	31	61	89	107	127	521	607	1279	2203	2281
f	1933	149	139	1609	83	23	47	373	191	23

The thing most interesting about this and similar conjectures is whether they can be disproved without resorting to extended numerical computations. The factors here listed were found using the SWAC, but, once discovered, they are easily verified by elementary computation.

What of the conjecture that, when $p>19$ and 2^p-1 is prime, then 2^p+99 is always composite?

Also solved by H. F. Bennett, Sidney Kravitz, D. C. B. Marsh, R. E. Shafer, Alan Wayne, and the proposer.

A Limit of a Summation

4684 [1956, 258]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Prove that $a_n \to 0$ as $n \to \infty$, where

$$a_n = 1 - \frac{n-1}{1!} + \frac{(n-2)^2}{2!} - \frac{(n-3)^3}{3!} + \cdots + \frac{1}{(n-1)!}.$$

Solution by Daniel Shanks, Naval Ordnance Laboratory, Silver Spring, Md. If

$$f(z) = (e^z - z)^{-1} = e^{-z}(1 - ze^{-z})^{-1} = e^{-z} + ze^{-2z} + z^2e^{-3z} + \cdots$$

is expanded into the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\sum a_n$ converges and $a_n \to 0$ as desired, provided every singularity of $f(z)$ satisfies $|z| > 1$.

Now if $z = e^z$ with $|z| \leqq 1$ we will have

$$x + iy = e^{x+iy} = e^x(\cos y + i \sin y), \qquad x^2 + y^2 \leqq 1,$$

so that $y^2 < (\pi/2)^2$ and $x = y/\tan y > 0$. Thus $|z| = e^x|e^{iy}| = e^x > 1$. This contradiction completes the proof.

Also solved by Leonard Carlitz, Peter Henrici, W. J. Pervin, H. S. Shapiro, and the proposer.

The n-dimensional Polytope

4685 [1956, 258]. *Proposed by L. J. Mordell, University of Toronto*

Prove that the n-dimensional polytope

$$(1) \qquad |x_1| + |x_2| + \cdots + |x_n| + |x_1 + x_2 + \cdots + x_n| \leqq 2,$$

has $2^{n+1} - 2$ faces of $n-1$ dimensions and contains an n-dimensional volume $(2n)!/(n!)^2$.

Solution by Jože Ulčar, Institute of Mathematics, Skoplje, Yugoslavia. The equation (1) defines a convex polytope, since every line $x_i = k_i t$ $(i = 1, 2, \cdots, n)$ intersects its boundary in two points which are symmetric to the origin and have coordinates $x_i = \pm 2k_i/(\sum|k_i| + |\sum k_i|)$.

1. The coordinate hyperplanes separate the Euclidean space R_n into 2^n parts. In each of the parts where all coordinates do not have the same sign, there lie two $(n-1)$-dimensional faces of the polytope (1). For example, in that part where $x_{r_1}, x_{r_2}, \cdots, x_{r_i}$ are positive and $x_{r_{i+1}}, x_{r_{i+2}}, \cdots, x_{r_n}$ are negative, they are the faces

$$(2) \qquad x_{r_1} + x_{r_2} + \cdots x_{r_i} = 1, \qquad x_{r_{i+1}} + x_{r_{i+2}} + \cdots + x_{r_n} = -1.$$

In the part in which all x_i are positive, the polytope (1) has only one $(n-1)$-dimensional face, namely

$$(3) \qquad x_1 + x_2 + \cdots + x_n = 1,$$

and in the part in which all x_i are negative, only the face

$$x_1 + x_2 + \cdots + x_n = -1. \tag{4}$$

Altogether, then, the number of $(n-1)$-dimensional faces of the polytope (1) is $(2^n-2)\cdot 2+2=2^{n+1}-2$.

2. The polytopes which are bounded by the coordinate hyperplanes and by either of the hyperplanes (3) and (4), are simplexes. Each has the volume $1/n!$.

The polytope which is bounded by the coordinate hyperplanes and the hyperplanes (2) is a simplotope* whose n-dimensional volume is equal to the product of the i-dimensional volume and the $(n-i)$-dimensional volume determined respectively by the hyperplanes (2) and the coordinate hyperplanes; hence the volume is $(1/i!)(1/(n-1)!)$. Since there are $\binom{n}{i}$ parts of R_n for which i coordinates are positive, the volume of the entire polytope (1) equals

$$V = \sum_{i=1}^{n-1} \frac{1}{i!(n-i)!}\binom{n}{i} + \frac{2}{n!} = \sum_{i=0}^{n} \frac{1}{i!(n-i)!}\binom{n}{i} = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}^2.$$

This last form for V reduces to the proposed form upon observing that

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}.$$

Two Summations

4686 [1956, 258]. *Proposed by Leonard Carlitz, Duke University*

Show that

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n}(2k-1)\tanh\frac{(2k-1)\pi}{2} - n^2\right\} = -1/12. \tag{1}$$

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\coth k\pi - \frac{n(n+1)}{2}\right\} = 1/12 - 1/4\pi. \tag{2}$$

Solution by Peter Henrici, University of California, Los Angeles. In view of

$$\tanh x = 1 - 2/(e^{2x}+1), \qquad \coth x = 1 + 2/(e^{2x}-1),$$

the stated relations are equivalent to

$$2\sum_{k=1}^{\infty}\frac{2k-1}{e^{(2k-1)\pi}+1} = \frac{1}{12}, \qquad 2\sum_{k=1}^{\infty}\frac{k}{e^{2k\pi}-1} = \frac{1}{12} - \frac{1}{4\pi}.$$

But these are known identities; see, e.g., H. F. Sandham, *Some infinite series*, Proc. Amer. Math. Soc. vol. 5, 1954, pp. 430–436, equations (4.22) and (6.221). See also Problem 4453 [1952, 706].

Also solved by J. R. Hatcher, R. G. Stoneham, Chih-yi Wang, and the proposer.

* See H. S. M. Coxeter, Regular Polytopes, New York and Chicago, 1948, footnote p. 124.

A Double Summation

4693 [1956, 426]. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!n!}{(m+n+2)!}=\frac{\pi^2}{6}.$$

Solution by J. S. Frame, Michigan State University. Multiplying each term by $(m+n+2)-(n+1)$ and dividing by $(m+1)$, the summation over n can be written as a telescoping series, and the double sum becomes

$$\sum_{m=0}^{\infty}\frac{m!}{m+1}\sum_{n=0}^{\infty}\left(\frac{n!}{(m+n+1)!}-\frac{(n+1)!}{(m+n+2)!}\right)$$

$$=\sum_{m=0}^{\infty}\frac{m!}{m+1}\,\frac{0!}{(m+1)!}=\sum_{m=0}^{\infty}\frac{1}{(m+1)^2}=\frac{\pi^2}{6}.$$

Also solved by R. P. Agnew, Fred Brafman, K. A. Bush, Leonard Carlitz, P. L. Chessin, Leopold Flatto, M. L. Frimer and A. L. Tritter, R. K. Guy, J. R. Hatcher, Peter Henrici, S. S. Holland, Jr., W. C. Janes, J. B. Johnston, M. A. Kirchberg, P. G. Kirmser, M. S. Knebelman, W. R. Knight, A. E. Livingston, Marshall Luban, Y. L. Luke, D. C. B. Marsh, Kovina Milosevich, D. R. Morrison, Ingram Olkin, N. J. Phillips, E. J. F. Primrose, Chih-yi Wang, R. E. Wild, Henri Yerly, David Zeitlin, Judith Zoffmann, and the proposer. Late solutions by Žiradin Pantić and J. R. Trollope.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Engineering Mathematics. By Kenneth S. Miller. Rinehart & Co., New York, 1956. xii+417 pp. $6.50.

This text is intended for the first year of graduate work for the engineering student with "a little college algebra, analytics, calculus, and a smattering of differential equations." It assumes a more substantial maturity, however, for it covers a wide selection of topics, with a thoroughness which should lighten the burden of teaching. In perusing this exposition one finds many pet teaching devices incorporated into the text. Although the author seems anxious to argue that each topic introduced is necessary for a coherent presentation, it seems that the various sections are largely independent in treatment, so that, by proper

selection, the book could be adapted to a variety of courses, including the post-calculus course for undergraduates which has become popular in recent years. There is ample material for two four-semester-hour undergraduate courses, or six hours on the graduate level.

The first chapter proves the basic theorems on determinants, but there is only a brief glimpse of matrices. Another preliminary chapter is devoted to functions defined by integrals such as gamma, beta, sine-integral, and elliptic functions, and such topics as Jacobians, Stirling's formula and Euler's constant, are concisely treated. Breezing through a review of linear differential equations with constant coefficients, the author then gives a careful exposition of the method of variation of parameters leading to the Green's function. The method of Frobenius is given a heuristic treatment, showing how the coefficients may be successively computed, and postulating the convergence of the series by quoting an appropriate existence theorem. The Legendre and Bessel equations are solved, but their derivation from Laplace's differential equation in spherical, and cylindrical coordinates is implied, but not explicitly shown. Fourier series and integrals are derived and used. The Laplace transform chapter includes the convolution theorem. The unit on network theory includes many engineering applications. The section on statistics seems more advanced than the others, and it is questionable whether an engineering student could benefit by this study of stochastic and ergodic processes without additional pre-requisites. (The algebra and calculus of vectors is not included.)

The author asserts that "the level of rigor is at the presently accepted standard for engineering schools." This apparently means that limiting processes are interchanged with the validity of the answer accepted as sufficient justification. This reviewer would take exception to such statements as: "Given the function $\int_0^x e^{-x^2}dx$, one cannot find a function $\psi(x)$ such that $\psi'(x)=e^{-x^2}$." But though one may quibble over wording, or the motivation for introducing the successive topics, the expositions of the steps necessary to solve the type of problems selected are careful and explicit. The exercises are relatively few but no student could work all of them. The exposition is aided by a number of well-chosen figures, and each chapter terminates with a bibliography. Three appendices, answers, and an index are included in this volume.

Any teacher of engineering mathematics will enjoy reading this compact book.

ARTHUR BERNHART
University of Oklahoma

A Portrait of 2. By L. A. Ringenberg. Washington, National Council of Teachers of Mathematics, 1956. 42 pp. $.75.

This is not, as the title might imply, a discussion of the special properties of the integer 2. Rather it is a development of the number system of algebra from the concept of ordinal and cardinal numbers through the complex numbers, with

2 used as an example to illustrate the evolving generalizations. Thus we first see 2 as an ordinal number and then as a cardinal number. Next $+2$ is seen as the class of all ordered pairs of natural numbers (a, b) for which $a-b=2$ and it is pointed out that, contrary to the average beginning algebra student's ideas (and even to conventional mathematical notation), 2 (the natural number) and $+2$ (the integer) are not identical but are related through the concept of isomorphism.

A similar type of discussion lets us see a view of 2 as a rational number, as a real number (both as a Dedekind cut and as a Cantor class of sequences), and as a complex number.

While similar developments of the number systems of elementary algebra may be found in many places, few are as detailed and as motivated as this one and none, to my knowledge, bring the discussion to bear so closely on the high school teacher's problems (as, for example, in the author's discussion of the dual role of the $+$ sign). Although the treatment is kept on as elementary a level as possible necessary technicalities are not left out. Thus, for example, it is clearly pointed out that $\sqrt{2}$ is not just a sequence of rational numbers but, rather, a *class* of sequences. Although the various proofs involved in the development are not generally given, it is made clear what proofs are needed and, in most cases, how they might be carried out. Suggestions for further readings are also given.

If every high school teacher were to read and master the contents of this little book the level of instruction in high school algebra should rise considerably.

ROY DUBISCH
Fresno State College

Numerical Analysis. (Proceedings of Symposia in Applied Mathematics, vol. 6). Edited by John H. Curtiss. McGraw-Hill, New York, 1956, for the American Mathematical Society. vi+303 pp. $9.75.

In 1943 a proposal to hold a symposium on numerical analysis would probably have met with consternation, to say the least, especially if the proposal were made to the American Mathematical Society. The expression itself would have been unfamiliar, for the Institute for Numerical Analysis had not yet come into existence. However, if one could imagine such a symposium being held, it might have been sparked by the table makers of the Bureau of Standards (formerly a WPA project), and by the ballisticians of Aberdeen, and perhaps Dahlgren. Doubtless the astronomers would have been invited along with the geodesists and map makers, and perhaps actuaries and statisticians. Topics would have included the calculation of trajectories and orbits, least square fitting and smoothing of data, interpolation and numerical quadrature, and perhaps the solution of nonlinear equations. Methods of relaxation for solving problems in elasticity and electrostatics might have come in for some discussion, with a bow to the engineers. There may or may not have been discussion

of characteristic roots and vectors, since who would have had anything novel to contribute? And almost certainly the trivial problem of solving linear algebraic systems would not have come up unless someone wanted to offer a new variant of the Doolittle method. If the scope were broadened slightly to bring in the psychometrists, they could have talked about principal axes, rotations, and simple structure, concepts related to positive definite symmetric matrices. But the objectives here would often have been poorly defined, hence the techniques fuzzy, and most of the audience might have felt more confused than enlightened. If a scattering of physicists and chemists had attended, they would have been there mainly to listen, but the mathematicians would have been most conspicuous by their absence.

In 1943 there was some use of punched card machines by the astronomers, but for the most part desk computers were the most elaborate computing aids then in use. The Mark I, the ENIAC, and the relay machines of the Bell Laboratories were far from complete. The literature on computing was scattered and concerned with special techniques, some of which were repeatedly rediscovered and described in varied settings, except in the areas of interpolation and finite differencing, which gave rise to and bordered on a well elaborated mathematical theory. Elsewhere theory was sparse, in spite of the occurrence of such names as Gauss, Bernoulli, and Newton.

In 1953, the Mark I, the ENIAC and the Bell relay machines were already obsolete; the SEAC was to begin its fourth year of operation, Univacs were appearing, and many other electronic digital machines were at varying stages of development. The Institute for Numerical Analysis, set up under the National Bureau of Standards, had brought together outstanding mathematicians and outstanding professional computers and encouraged the introduction of mathematical rigor into the evaluation of numerical techniques. Much of the support for this and for other related projects had come from the armed services, especially the Office of Naval Research.

In this atmosphere the sixth Symposium in Applied Mathematics, arranged by the American Mathematical Society, was held at Santa Monica and devoted to numerical analysis. Of the twenty-one papers delivered, nineteen are published in the volume here under review. Contributors are exclusively mathematicians and physicists. Nowhere in either the index or the table of contents is there mention of orbits, trajectories, finite differences, or smoothing, and only two references are made to interpolation. However, there are three papers dealing with the solution of linear algebraic systems, and a fourth dealing with those special systems which appear as partial difference equations.

Broadly, the topics fall into two classes, techniques and applications, although most papers give some consideration to both. The applications are interesting. One paper by Clutterham and Taub, deals with a problem in hydrodynamics, but there is at most passing reference to specific problems in electrostatics, elasticity, or other phases of classical applied mathematics. On the other hand, there are three papers, by Bruck, Emma Lehmer, and Taussky, on ap-

plications to number theory, a paper by Bellman on dynamic programming, one by Motzkin on the assignment problem, and one by Tompkins on problems whose variables are permutations. In 1943, dynamic programming had not been invented, the assignment problem had been stated as a mathematical problem but without recognized practical application, the problems discussed by Tompkins would probably have been regarded as without interest. As for the number theorists, they had for some time resorted to computing for suggesting and testing conjectures, but these efforts were not commonly reported. It is the electronic computer that makes the difference.

Generally speaking, the papers are all at an advanced level, and the methodological papers especially so. Bergman discusses boundary value problems of linear partial differential equations. The longest paper of all is by Rosenbloom on the method of steepest descent, in quite abstract terms. The method has a host of applications. Sard describes his method of obtaining remainder terms for functional approximations, Walsh discusses *Best-approximation Polynomials of Given Degree*, and the Hastings approximations, based upon the Chebyshev theory, are discussed by Hastings. Particular mention should be made of a paper by Wielandt entitled *Error Bounds for Eigenvalues of Symmetric Integral Equations*, since it deals with assessments in a situation where few useful results are available.

Until a very few years ago only two distinct methods of solving linear systems were in common use. One of these is direct, the other iterative, and both go by the name of Gauss. The so-called Doolittle method is a particular form of Gaussian elimination. A number of new methods are now known with varying degrees of distinctiveness, but perhaps the most significant is the method of conjugate gradients, developed independently by Stiefel and by Hestenes. In this volume, Hestenes discusses the conjugate gradient method, and Fischbach discusses "gradient methods," including the method of conjugate gradients. A third paper on linear systems, by David Young, discusses iterative methods, with special reference to linear difference equations. Frankel discusses linear partial difference equations, but confines himself to discursive remarks on the stability problem.

Only two papers remain to be mentioned. These are by Wasow on transforming distributions, and by Warschawski on conformal mapping.

Evaluation of such a volume as this should be in terms of what it contains. It should not be condemned for its omissions. But for purposes of orientation some of the more obvious ommissions might be mentioned. Ordinary differential equations and nonlinear systems of algebraic and transcendental equations are considered only indirectly, in the sense that the method of steepest descent can be applied to both. Certain aspects of partial differential equations come in for consideration by Bergman, Frankel and Young, but the treatment is very limited. Considerably more could be said on the solution of linear systems, and the characteristic value problem for matrices is not discussed at all except indirectly by Wielandt. Finally, there is only passing reference to the Monte Carlo

method.

Thus the book is not one for the novice who wishes a uniform development of numerical analysis in its present state. Neither is it a book for the practicing computer who wishes to find a collection of special techniques to be applied routinely. Its greatest appeal will be to the mathematician who would like to learn about some of the mathematical problems that arise in this important field. Such a person, whether or not he has had experience in computing, will find much to interest him.

A. S. HOUSEHOLDER
Oak Ridge National Laboratory and
Mathematics Research Center
United States Army

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER PROGRAM OF CANADIAN MATHEMATICAL CONGRESS

A joint Summer Seminar of the Canadian Mathematical Congress and the Theoretical Physics Division, Canadian Association of Physicists, will be held at the University of Alberta, Edmonton, August 12–30, 1957.

Congress research lectures will be given by the following: A. D. Alexandrov, University of Leningrad; Philip Hall, Cambridge University; D. H. Lehmer, University of California, Berkeley; Jean Dixmier, Institut Henri Poincaré; E. P. Wigner, Princeton University.

Congress instructional lectures are as follows: Leo Moser, University of Alberta, introduction to the theory of numbers; Abraham Robinson, University of Toronto, an introduction to mathematical logic; A. W. Tucker, Princeton University, theory of games; H. J. Zassenhaus, McGill University, introduction to Lie algebras.

A joint Congress of the two organizations will be held at the School of Fine Arts, Banff, Alberta, September 1–7, 1957.

Inquiries about the Seminar and the Congress should be addressed to Canadian Mathematical Congress, Secretariat, Chemistry Building, McGill University, Montreal, Quebec.

Inquiries regarding accommodation at Edmonton should be sent to Professor E. S. Keeping, Department of Mathematics, University of Alberta, Edmonton, Canada; and at Banff, to Professor D. R. Crosby, Department of Mathematics, or to Professor D. D. Betts, Department of Physics, University of Alberta, Edmonton, Canada.

MEMBERSHIP PROGRAM OF INSTITUTE OF MATHEMATICAL SCIENCES

The Institute of Mathematical Sciences, New York University, offers temporary memberships to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is active. These fields

include functional analysis, ordinary and partial differential equations, mathematical physics, fluid dynamics, electromagnetic theory, numerical analysis and digital computing, and various specialized branches, such as hydromagnetics and reactor theory.

The temporary membership program is designed primarily as a means of alleviating the present critical shortage of scientists trained in mathematical physics, applied mathematics, and related fields of mathematical analysis. The program is being supported by the National Science Foundation and also by funds contributed by industrial firms.

Temporary members may participate freely in the research projects, the advanced graduate courses and the research seminars of the Institute, and they will have the opportunity of using the computational facilities.

The temporary members will receive a stipend commensurate with their status. Membership will be awarded for one year, but it may be renewed. Special arrangements can be made for applicants who expect to be on leave of absence from their institutions.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, New York.

NATIONAL HIGH SCHOOL AND JUNIOR COLLEGE MATHEMATICS CLUB

A National High School and Junior College Mathematics Club has been formed. It is anticipated that high school mathematics clubs, if properly qualified, will join the new organization. The purpose of the club is to engender keener interest in mathematics, to develop sound scholarship in the subject and promote enjoyment of mathematics among high school and junior college students. For further information write to Secretary-Treasurer, Josephine P. Andree, Box 1127, University of Oklahoma, Norman.

Other officers are: President, H. L. Alder, University of California, Davis; Vice-President, E. L. Walters, William Penn Senior High, York, Pennsylvania; Governors-General, Nellie M. Kitchens, Hickman High School, Columbia, Missouri; Dr. J. R. Mayor, Director, Science Teaching Improvement Program, Washington, D. C.; Virginia L. Pratt, Central High School, Omaha, Nebraska.

CONFERENCE ON CAREERS IN THE MATHEMATICAL SCIENCES

A conference on Careers in the Mathematical Sciences was held by the Institute of Mathematical Sciences, New York University, on January 5, 1957. High school mathematics chairmen and vocational guidance counselors from New York, New Jersey, and Connecticut participated in the conference.

The program included a discussion of employment opportunities at various levels, a description of some of the work in which mathematicians are involved, and a consideration of the role of high-speed electronic computing machines in science and technology. The speakers were: Professor Richard Courant, Dr. L. W. Cohen, Dr. C. R. DeCarlo, Professor Harold Grad, Professor Morris Kline, President C. V. Newsom, Dr. Mina S. Rees, Dean H. W. Stoke, and Professor J. J. Stoker.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1957.

Columbia University, Teachers College, July 7 to August 16: Professor Rosskopf, department seminar in teaching mathematics; Dr. Sobel, professionalized subject matter in advanced secondary school mathematics, materials and models in mathematics education; teaching of geometry; mathematics in the junior high school. Under the National Science Foundation: Professor Levi, sets, algebra and analysis; Professor Lott, Jr., probability and statistics; Professor Rosskopf and Mr. Rourke, teaching modern concepts in the secondary school.

DePaul University, June 24 to August 3: Professor DeCicco, abstract metric geometry, abstract set theory; Mr. Merkes, elementary differential geometry; Dr. G. Weiss, topics in conformal mapping.

Kent State University, June 17 to July 20: Professor Iwanchuk, selected topics for elementary school teachers; Professor Bush, introduction to modern algebra; Professor Jenkins, introduction to mathematical statistics. July 22 to August 24: Professor Dressler, solid analytic geometry; Professor Johnson, history of mathematics; Professor Stapleford, advanced methods of teaching mathematics in high school.

Michigan State University, June 25 to August 2: Professor Blair, differential equations, functions of a complex variable; Professor Faith, college geometry; Professor Gaddum, advanced calculus III; Professor Nordhaus, potential theory; Professor Powell, concepts in mathematics (this course is part of a summer institute in mathematics, physics and chemistry sponsored by the National Science Foundation). June 25 to August 23: Professor Campbell, differential equations; Professor Herzog, functions of a complex variable III; Dr. Larcher, advanced mathematics for engineers II; Professor Oehmke, introduction to higher algebra III; Professor Parkus, advanced mathematics for engineers III, mathematical theory of elasticity, gas dynamics; Professor Peirce, vector analysis, computer coding I; Professor Reid, advanced calculus I, differential equations II; Professor Weeg, theory of matrices and groups, finite differences.

New York University, Institute of Mathematical Sciences, June 17 to July 26: Dr. Rubin, advanced calculus and applications I; Research Professor Magnus, special functions of mathematical physics; Professor Shapiro, special topics in linear programming (all half-courses). July 29 to September 6: Professor Shapiro, special topics in number theory; Staff, advanced calculus and applications II, Laplace transform and Heaviside calculus (half-courses).

Northwestern University, June 22 to August 17: Engineering mathematics I, III; numerical methods in mathematics; advanced calculus; geometry for teachers; complex variables for applications; topics in modern mathematics for teachers; modern algebra.

Stanford University, June 25 to August 18: differential equations; introduction to the functions of a complex variable; selected topics from analysis; graduate seminar; Professor Herriot, computer laboratory; Staff, advanced reading and research.

University of Buffalo, July 1 to August 9: Professor Bienert (Hobart College), finite differences, topics in geometry; Professor Gehman, history of mathematics; Professor Schneckenburger, theory of sets.

University of Chicago, June 24 to August 30: Program emphasizes algebraic topology; Professor MacLane, introduction to algebraic topology; Professor Spanier, homology of locally compact spaces; Professor Eilenberg (Columbia University), foundations of fiber bundles; Professor Thom (University of Strassbourg, France), homotopy of functional spaces. Other advanced courses: Professor Chern and Kuranishi, infinite continuous pseudo-groups; Professor Segal, analysis in topological groups; Staff, advanced calculus; algebra; projective geometry; set theory; complex variables.

University of Florida, June 14 to August 10: Professor Blake, introduction to mathematical thought; Professor South, mathematical statistics; Professors Lang, Blake, and Rohde, higher mathematics for engineers and physicists; Professor Phipps, foundation of geometry; Professor Cowan, Fourier series; Professor Kokomoor, history of elementary mathematics; Professor Hadlock, advanced topics in calculus; Professor Moore, theory of groups of finite order; Professor Pirenian, vector analysis; Professor Hutcherson, synthetic projective geometry; Professor Butson, general topology.

University of Illinois, June 17 to August 10: Professor Munroe, functions of real variables; Professor Hamilton, elementary geometry from a modern viewpoint; Dr. Zaring, group theory; Professor Landin, the teaching of selected mathematical topics in secondary schools.

University of Michigan, June 24 to August 16: Professor Coburn vector analysis;

Professor Coxeter, college geometry, higher geometry; Professor Craig, probability, theory of statistics II; Professor Dushnik, advanced calculus; Professor Harary, foundations of mathematics; Professor Hewitt, functions of real variable, abstract harmonic analysis; Professor Higman, theory of matrices, Galois theory; Professor Jones, teaching of algebra, history of elementary collegiate mathematics; Professor Rainville, intermediate differential equations; Professor Rothe, operational mathematics, integral equations; Professor Shields, Fourier series, dimension theory; Professor Titus, advanced calculus for engineers; Professor Ullman, functions of complex variable with applications; Professor Wesler, calculus of finite differences, theory of statistics I.

University of Minnesota, College of Science, Literature and Arts, June 17 to July 20: Professor Gibbens, solid analytic geometry; Professor Shapiro, probability, topics in topology. July 22 to August 24: Professor Olmsted, introduction to the theory of functions, advanced algebraic theory; Professor Munro, theory of numbers, calculus of finite differences. An Institute for High School Teachers of Mathematics will be conducted from June 17 to August 10.

University of Nebraska, June 13 to August 2: Dr. Cree, differential equations; Dr. Schweppe, projective geometry; Professor Leavitt, theory of equations; Professor Basoco, Laplace transforms; Staff, reading course, thesis.

University of North Carolina, June 6 to July 13: Professor Mackie, theory of equations; Professor Linker, differential equations; Professor MacNerney, partial differential equations, topics in analysis. July 15 to August 21: Professor Hoyle, elementary algebra from an advanced viewpoint; Professor Garner, introduction to higher geometry; Professor Hill, elementary mathematical statistics; Professor Lasley, analytic geometry from a higher standpoint; Professor Wall, some recent results in algebra.

University of Oklahoma, June 6 to August 3: Professor Bernhart, college geometry; Professor Springer, elementary differential equations, ordinary and partial differential equations; Professor LaFon, vector analysis; Professor Brixey, theory of groups.

University of Pennsylvania, June 24 to August 3: Professor Gottschalk, axiomatics of number systems; Professor Epstein, numerical methods and differential equations, theory of functions of a complex variable; Professor Yang, introduction to higher algebra.

University of Pittsburgh, June 10 to July 19, July 22 to August 30: Professor Barsotti, differential equations; Professors Blumberg and Christiano, advanced calculus; Professor Taylor, functions of a complex variable; Professor Bryson, partial differential equations and Fourier series; Professor Bompiani, algebraic geometry; Professor Laush, calculus of variations; Professor Levine, topology. June 17 to August 9 (evenings): Professors Edwards and Knipp, differential equations; Professors Bryson and Laird, mathematics of modern engineering; Professor Laush, integral equations; Professor Levine, functions of a complex variable (second half of two semester course). July 1 to August 9: Professor Elyash, differential equations for students of engineering; Professor Teats, history of mathematics; Professor Kehl, introduction to the mathematics of digital computation; Professor Myers, recreational mathematics for teachers, teaching of secondary school mathematics; Professor Kachun, navigation for teachers; Professor DeSua introduction to the foundations of mathematics.

University of South Carolina, June 10 to August 10: Professor Lee, theory of equations; Professor Novak, college geometry; Professor Williams, advanced calculus, theory of functions of complex variables; Professor Hedberg, introduction to modern algebra. A Summer Institute for Teachers of High School Mathematics will be held simultaneously with the Summer Session; Director, Professor W. L. Williams.

University of Texas, June 4 to July 16: Professor Craig, applications of tensor analysis; Professor Moore, theory of sets; Professor Ettlinger, research in integral equations. July 17 to August 27: Professor Lane, continued fractions; Professor Wall, analytic functions, infinite processes.

University of Virginia, June 24 to August 17: Professor Plunkett, foundations of

algebra (for 6 weeks beginning July 8), differential equations and applied mathematics; Professor Botts, introductory analysis; Professor Floyd, analytic topology.

University of Washington, June 24 to August 22: Professor Walter, linear algebra; Professors McFarlan, Kingston, and Leipnik, differential equations; Professor Avann, advanced calculus and vector analysis, applications of vector analysis; Professor Cramlet, topics in applied analysis; Professor Livingston, advanced euclidean geometry; non-euclidean geometry; Professor Pierce, foundations of mathematics.

University of Wyoming, June 10 to July 12: Professor Walsh, theory of determinants and matrices; Staff, ordinary differential equations; Professor Barr, seminar in analysis; Professor Neubauer, history of mathematics. July 15 to August 16: Professor Varineau, advanced calculus, fundamental concepts of mathematics; Professor S. R. Smith, partial differential equations, Fourier series and boundary value problems.

Virginia Polytechnic Institute, Southern Regional Graduate Summer Session in Statistics, June 12 to July 20: Dr. DeLury (Ontario Research Foundation, Toronto, Canada) and Dr. McHugh (Virginia Fisheries Laboratory), sampling of biological populations; Dr. Williams (Commonwealth Scientific and Research Organization, Melbourne, Australia), analysis of variance; Professor Ash, statistical inference; Professor Brenna, engineering statistics; Professor Bradley, rank order statistics; Professor Clunies-Ross, probability; Professor J. E. Freund, stochastic processes; Professor R. J. Freund, sampling; Professor Kramer, statistical methods; Professor Wine, theory of least squares. Director: Professor Boyd Harshbarger.

Wayne State University, Computation Laboratory, June 3 to 8: introduction to computers and their applications. June 10 to 15: data processing in business and industry. September 9 to 14: industrial and management computer applications. Experts from various parts of the country and the staff of the laboratory will participate. A Conference on Matrix Computations will be held September 3 to 6 (see May MONTHLY).

PERSONAL ITEMS

Professor J. M. Barbour, Michigan State University, has been elected President of the American Musicological Society for the term 1957–59.

Professor Emeritus George Polya, Stanford University, has been elected to honorary membership in the London Mathematical Society.

College of Arts and Sciences, Baghdad, Iraq: Professor H. Schatz has been appointed Professor; Dr. M. W. Al-Dhahir has been promoted to Assistant Professor; Mr. Waleed Al-Salam has been given a scholarship to study at Duke University; Mr. Y. Al-Duri is studying at the University of Oregon.

Michigan State University: Dr. Heinrich Larcher, Institute Montana, Zugerberg, Switzerland, has been appointed Instructor; Dr. W. H. Peirce, University of Wisconsin, has been appointed Assistant Professor; Professor Henry Parkus, Institute of Technology, Vienna, has been appointed Visiting Professor for the spring and summer terms of 1957; Dr. S. K. Berberian and Dr. J. H. McKay have been promoted to Assistant Professors.

University of Virginia: Dr. R. P. Goblirsch, Teaching Assistant, University of Wisconsin, has been appointed Instructor; Associate Professor E. E. Floyd has been promoted to Professor.

Mr. A. T. Bharucha-Reid has been appointed Instructor at the University of Oregon.

Associate Professor O. C. Collins, University of Nebraska, is on leave of absence and is a resident consultant with General Mills, Minneapolis, Minnesota.

Dr. Martin D. Davis, Ohio State University, has been appointed Assistant Professor at Hartford Graduate Center, Rensselaer Polytechnic Institute.

Dr. D. O. Ellis, Electronics Division, National Cash Register Company, Hawthorne, California, has a position as a research scientist and head of the analysis staff of Litton Industries, Beverly Hills, California.

Assistant Professor R. L. Gay, Wake Forest College, has been promoted to Associate Professor.

Dr. W. A. Howard, University of Chicago, has accepted a position as a mathematician with Bell Telephone Laboratories, Whippany, New Jersey.

Mr. O. C. Juelich is employed now as a computing engineer at North American Aviation, Columbus, Ohio.

Dr. J. Lawrence Katz, Polytechnic Institute of Brooklyn, has been appointed Assistant Professor of Physics at Rensselaer Polytechnic Institute.

Dr. Manfred Kochen, Columbia University and the Institute for Advanced Study, has accepted a position as an associate mathematician with the International Business Machines Corporation Research Center, Poughkeepsie, New York. Dr. Kochen was formerly a Ford Foundation fellow at Harvard University.

Associate Professor J. A. Larrivee, Worcester Polytechnic Institute, is employed as a mathematical analyst by Lockheed Aircraft Corporation, Burbank, California.

Dr. W. S. Mahavier, Defense Research Laboratory, University of Texas, has been appointed Instructor at Illinois Institute of Technology.

Mr. H. W. Martin, Clarendon Junior College, has been appointed Instructor at Amarillo College.

Miss Ethelyn L. McBee, University of Florida, has been appointed Associate Professor and Head of the Department of Mathematics at Wesleyan College.

Mr. R. P. Mitchell, White Sands Signal Corps Agency, has accepted a position as a mathematician with the Naval Ordnance Laboratory, Corona, California.

Professor C. N. Moore, University of Cincinnati, is a visiting lecturer at the University of South Carolina for the second semester of 1956–57.

Dr. W. L. Nicholson, Princeton University, has accepted a position as a research statistician with General Electric Company, Richland, Washington.

Mr. R. J. Pipino has been appointed Instructor at Western Reserve University.

Mr. G. F. Simmons, Yale University, has been appointed Assistant Professor at the University of Rhode Island.

Mr. W. W. Varner, Head, Computation Laboratory, University of Colorado, has a position as Head of the Flight Simulation Laboratory and Computers, Convair Astronautics, General Dynamics Corporation, San Diego, California.

Miss Marion I. Walter, Teaching Assistant, Cornell University, has been appointed Assistant Professor at Simmons College.

Professor Emeritus E. P. Adams of Princeton University died on December 31, 1956. He was a charter member of the Association.

Dr. Tobias Dantzig, who had retired from the position of Chairman of the Department of Mathematics, University of Maryland, died on August 9, 1956.

Professor F. W. Reed of Ohio University died on November 22, 1956. He was a member of the Association for thirty-seven years.

Professor Emeritus Carrie B. Taliaferro of State Teachers College, Farmville, Virginia, died on September 14, 1955. She was a member of the Association for thirty-four years.

Professor Emeritus H. S. Uhler of Yale University died on December 6, 1956. He was a charter member of the Association.

Professor Emeritus F. B. Wiley of Denison University died on December 14, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1956

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1956. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

There was a deficit of $4,111 in the Current Fund of the Association for the year 1956. This deficit was anticipated by the Finance Committee when the Budget for 1956 was adopted. It is believed that with increased receipts for dues and other items, the Current Fund will have no deficit during 1957.

Assets of the Association	January 1, 1956	December 31, 1956
M & T Trust Co., Buffalo	12,842.52	30,857.02
Buffalo Savings Bank	20,000.00	19,297.28
Securities	123,832.00	111,828.64
	156,674.52	161,982.94
Funds of the Association		
Current Fund	2,561.37	450.03
Carus Fund	22,140.94	23,763.26
Chace Fund	13,839.25	12,369.13
Houck Fund	11,496.05	12,082.88
Chauvenet Fund	1,446.41	1,420.06
Dunkel Fund	18,954.24	19,917.92
General Fund	41,509.40	40,518.72
	111,947.66	110,522.00
Visiting Lecturers Fund	23,162.53	42,942.44
Fund for Committee on Undergraduate Program	21,564.33	8,069.77
Fund for Committee on Films	—	448.73
	156,674.52	161,982.94

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 171 persons have been elected to membership by the Board of Governors on applications duly certified.

R. J. Adler, Student, Carnegie Institute of Technology

S. J. Adler, Ph.D. (Columbia) Chemist, Brookhaven National Laboratory, Upton, N. Y.

William Allan, B.A. (Amherst) Actuary, Home Life Insurance Co., New York, N. Y.

Rev. R. W. Allen, S.J., Ph.D. (St. Louis) Chm., Department of Mathematics, Xavier University.

Yale Altman, M.S. (Boston U.) Field Representative, Addison-Wesley Publishing Co., Reading, Mass.

Maurice Anderson, A.B. (Wayne S.T.C.) Grad. Asst., University of Nebraska.

Mansur Arbabi, B.S. in E.E. (California State Poly. C.) Instr., Seattle University.

R. J. Arguello, B.E.E. (Poly. Inst. of Brooklyn) Electrical Design Engr., Otis Elevator Co., Brooklyn, N. Y.

J. W. Armstrong, A.B. (San Jose S.C.) Grad. Student, San Jose State College.

Lawrence Arnold, Student, Occidental College.

F. W. Ashley, Jr., B.S. (Oklahoma) Lexington, Massachusetts.

M. H. Barnebey, M.S. (Ohio S.U.) Asso. Professor, Tougaloo Southern Christian College.

G. D. Bisbey, M.S. (Iowa S.C.) Instr., Wisconsin State College, River Falls.

J. T. Bliss, B.S. (Black Hills T.C.) Math., Directorate of Research and Development, Holloman Air Development Center, N. Mex.

G. C. Branche, M.S. (Rochester) Grad. Asst., University of Rochester.

Rev. E. W. Brande, S.J., M.S. (St. Louis) Grad. Student, St. Louis University.

R. D. Branstetter, Ph.D. (Iowa S.C.) Asst. Professor, San Diego State College.

G. A. Brown, Grad. Student, Rutgers University.

A. J. Burda, Jr., B.S. (U. S. Naval Acad.) Instr., University of Maryland.

Mrs. Marjorie V. Butcher, M.A. (Michigan) Instr., Trinity College, Connecticut.

Jean S. Campbell, Math., Office of Naval Research, Washington, D. C.

D. L. Carlstrom, Student, Eastern Montana College of Education.

S. D. Chatterji, M.S. (Lucknow U.) Grad. Asst., Syracuse University.

Evan Chenet, Jr., M.A. (Columbia) New York, New York.

Rev. A. A. Clarke, S.J., M.A. (Fordham) Instr., Fordham University.

E. H. Connell, B.A. (McMurry) Res. Scientist, Missile Systems Division, Lockheed Aircraft Corp., Palo Alto, Calif.

H. H. Corson, Ph.D. (Duke) Instr., Tulane University.

H. L. Dachslager, M.S. (Wisconsin) Grad. Student, University of Wisconsin.

J. O. Danley, M.A. (Oklahoma) Instr., East Central State College.

C. T. Daub, Jr., Student, Carleton College.

H. J. Davis, M.S. (Stanford) Instr., Pomona College.

J. D. DePree, B.A. (Hope C.) Instr., University of Colorado.

R. L. Duncan, M.A. (Pennsylvania S.U.) Instr., Pennsylvania State University.

Janet M. Dunning, M.A. (Columbia U., Teachers C.) Head, Department of Mathematics, Ramapo Regional High School, N. J.

Philip Dwinger, Ph.D. (Leiden) Asst. Professor, Purdue University.

R. L. Edwards, Jr., B.E.E. (Alabama P.I.) Grad. Student, Johns Hopkins University.

Herman Elson, M.A. (N.Y.U.) Teacher, Technical High School, Buffalo, N. Y.; Instr., University of Buffalo.

Carlos Fallon, Lieutenant (Natiónal Military Acad., Colombia) Engr., Vitro Corporation of America, Silver Spring, Md.

H. L. Farris, Jr., B.A. (Texas) Math., Douglas Aircraft Co., Tulsa, Okla.

Walter Feit, Ph.D. (Michigan) United States Army.

Thorleif Fostvedt, M.S. (Chicago) Asso. Professor, University of Alberta.

Charles Fox, D.Sc. (London, England) Professor, McGill University.

L. B. Fuller, M.A. (Michigan) Grad. Asst., University of Rochester.

S. E. Ganis, M.S. (Michigan) Asso. Professor, Ohio Wesleyan University.

Dot J. Gifford, M.A. (Oklahoma) Asst. Professor, Oklahoma College for Women.

J. H. Gissel, M.S. (North Dakota) Instr., General Motors Institute, Flint, Mich.

I. I. Glick, B.A. (Johns Hopkins) Res. Asst., University of Maryland.

Esther E. Guerin, B.S. (Douglass) Teacher, Roxbury High School, Succasunna, N. J.

Mary Hagen, B.A. (Coll. of St. Elizabeth) Palisade, New Jersey.

D. L. Hartford, B.A. (Kentucky) Instr., University of Kentucky.

E. L. A. Hayes, B.S. (Seton Hall) United States Army.

J. W. Haynes, Jr., M.A. (California, Berkeley) Grad. Student, University of California.

Leon Henkin, Ph.D. (Princeton) Asso. Professor, University of California, Berkeley.

Mrs. Dagmar R. Henney, M.S. (Miami) Junior Instr., University of Maryland.

J. N. Herrmann, Res. Stress Analyst, Bendix Products Division, South Bend, Ind.

Aaron Herschfeld, M.A. (Columbia) Asst. Professor, Canisius College.

J. C. Hickman, M.S. (S.U. of Iowa) Actuarial Asst., Bankers Life Co., Des Moines, Iowa.

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THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the College of William and Mary, Williamsburg, Virginia, on December 1, 1956. Professor R. C. Yates, Chairman of the Section, presided at the morning and afternoon sessions.

There were 70 persons in attendance, including 40 members of the Association.

By unanimous vote of members present, the Section approved the following resolution:

Whereas the Mathematical Association of America has voted to sponsor a nationwide mathematics contest for high schools in 1958;

And, Whereas, the Md-DC-Va. Section has operated a successful local contest in these three states each year since 1954;

And, Whereas, certain advantages can be gained by coordination in contest activities, thus securing better support, prestige, and financial backing:

Therefore, be it resolved that the Md-DC-Va. Section does hereby—

(1) Express interest in participating in, and hereby offer its services on the National Standing Committee on High School Contests to be appointed by the President of the Association;

(2) Authorize its present contest committee to serve as the Sectional Committee on High School Contests, as mentioned in section IIIa (1) of the National Committee recommendations;

(3) Authorize this committee to continue to conduct our present contest program until transition to the national program is approved by our executive committee.

The following papers were presented:

1. *A unique formula for the ratio of the areas of two triangles, the one formed by transversals through the vertices of the other*, by Professor S. T. Gormsen, Virginia Polytechnic Institute.

A formula was developed which expresses the ratio of the area of a given triangle to the area of a second triangle, formed by passing three transversals through the vertices of the given triangle and dividing each of the opposite sides into two segments. As a consequence of this formula, proofs for at least four important geometric theorems about concurrency of lines, such as the converse of "Ceva's Theorem," follow immediately.

2. *The place of mathematics in Austrian education*, by Professor Herta T. Freitag, Hollins College.

Curricular offerings and teaching procedures on all levels of mathematics in the Austrian school system were sketched as experienced from 1914 to 1938, and observed in 1954.

3. *The geometry of the compass*, by Professor L. S. Shively, Bridgewater College.

This paper presented a brief exposition of the geometric constructions of Mascheroni (1797) in which the compass only is used. It was demonstrated that any construction which can be made with straight edge and compass can be made with the compass only. Also, the constructions with five fixed compasses, for the division of the circle into equal arcs, as made by Mascheroni, were given. One of these (division into 4 equal arcs) is the so-called problem of Napoleon.

4. *A theorem concerning the Bernstein polynomials*, by Mr. H. W. Gould, University of Virginia.

It was shown that if $f(x)$ is a polynomial of degree ν in x, then the Bernstein polynomials of $f(x)$ are given by $B_k^f(x) = f(x) + \sum_{\alpha=1}^{\nu} k^{-\alpha} Q_\alpha^\nu(x)$, where the $Q_\alpha^\nu(x)$ (independent of k) are polynomials of degree $\leqq \nu$ which can be given explicitly in terms of the Stirling numbers of the first and second kinds. The existence of $Q_\alpha^\nu(x)$ was proved by E. J. McShane, *Ann. of Math.*, Study 31, page 119.

5. *On the Gibb's phenomenon in the eigenfunction series associated with a non-self-adjoint differential equation*, by Professor L. I. Mishoe, Morgan State College.

This paper investigated the Gibb's phenomenon of the series expansion of an arbitrary function $f(x)$ in terms of the eigenfunctions of the system $(A+\lambda B)u=0$, $u(a)=u(b)=0$, where A is the differential operator $(d^2/dx^2)+q(x)$ and B is the operator $(d/dx)+p(x)$, and showed that such an expansion will exhibit the Gibb's phenomenon whenever the classical Fourier series does so.

6. *Contribution to the theory of the blunderbuss*, by Mr. Peter Treuenfels, Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Special solutions of the equations of nearly one-dimensional time dependent compressible isentropic flow of a perfect gas are studied, subject to the assumptions that the Mach number is independent of time and that the gas velocity varies inversely with time. The system of differential equations considered in this paper is non-singular at sonic velocity. The solutions can be completely described in terms of quadratures of elementary functions. Numerical values are presented. It is believed that the material would form a suitable topic of discussion for an undergraduate mathematics club.

7. *The National High School Mathematics Contest*, Panel Discussion by Professor R. P. Bailey, U. S. Naval Academy (Former Chairman, National Contest Committee), Mr. W. H. Norris, Norfolk Public Schools (Chairman, MD-DC-Va. Contest Committee), and Professor D. B. Lloyd, D. C. Teachers College (Former Chairman, NCTM Contest Committee).

The speakers discussed the implications of the National Contest Recommendations recently approved by the Board of Governors and considered the general question of the participation of the Section in the project.

8. *A plea for changed attitudes toward science*, by Dr. B. C. Dees, Assistant Director of Scientific Personnel and Education, National Science Foundation (invited address).

We must create a more favorable climate for science education in the United States. As we succeed in presenting science and mathematics to students and to the public in more meaningful, more accurate ways, we will create the kind of respect for science and scientists which will improve attitudes toward science. Clearly, we must train our high school teachers more adequately, and we must see to it that they are given more status and more pay. We must make sure that the right kinds of science curricula are made available to students at all educational levels—and that students are properly counseled so that they do not rule out for themselves careers in science by choosing unwisely the courses they take in high school.

R. P. BAILEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS, University of Kansas, Lawrence, April 13, 1957.

KENTUCKY, Berea College, Berea, April 27, 1957.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.

METROPOLITAN NEW YORK, Hunter College, New York, April 27, 1957.

MICHIGAN

MINNESOTA, Carleton College, Northfield, May 11, 1957.

MISSOURI, Southeast Missouri State College, Cape Girardeau, April 27, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.

NEW JERSEY, Fall, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 28, 1957.

NORTHERN CALIFORNIA, January, 1958.

OHIO, University of Cincinnati, April 20, 1957.

OKLAHOMA, University of Arkansas, Fayetteville, April 12–13, 1957.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.

PHILADELPHIA, November 28, 1957.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.

SOUTHEASTERN

SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April 26–27, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

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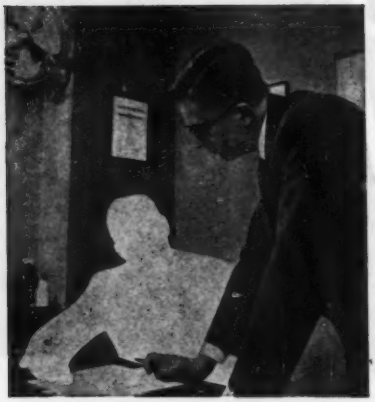

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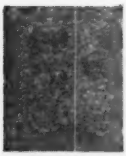

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